Determinacy and the Sharp Function on all sets

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Abstract. We characterize in terms of determinacy, the existence of the least inner model of "every set has a sharp." We also generalize our result.

Let $\#^\infty$ be the sharp function on all sets. We prove the "getting determinacy" direction (discovered independently by the Donald A. Martin and the author) of the equivalence between the existence of indiscernibles for $L\left[\#^\infty\right]$ and the determinacy of $\left(\Pi_1^1\right)_+^*$. We review the definition of $\left(\Pi_1^1\right)_+^*$, and more generally $\left(k*\Pi_1^1\right)_+^*$ for $k<\omega_1^{CK}$, which lie strictly between $\bigcup_{\beta<\omega^2}\left(\beta-\Pi_1^1\right)$ and $\Delta\left(\omega^2-\Pi_1^1\right)$.

Let $\#^{1\infty}=\#^\infty$. For $k<\omega_1^{CK}$, we set $\vec{\#}^{k\infty}=\langle\#^{(\beta+1)\infty}|\beta<k\rangle$ and define $\#^{(k+1)\infty}(x)$ so that it codes indiscernibles for $L(x)[\vec{\#}^{k\infty}]$. We prove the "getting determinacy" direction of the equivalence between the existence of indiscernibles for $L\left[\vec{\#}^{k\infty}\right]$ and the determinacy of $\left(k*\Pi_1^1\right)_+^*$.

Introduction. This is one of a series of papers in which we investigate the determinacy strength of the existence of indiscernibles for various least inner models of sharp functions. The classes whose determinacy we consider are all within $\Delta(\omega^2-\Pi_1^1)$ (see Definition 1.2). In Definitions 1.4 and 1.5, we review the definitions of $(\Gamma)^*$ and $(\Gamma)_+^*$. It is easy to show that $(\Pi_k^0)^*=(\Sigma_{k+1}^0)^*$ and $(\Pi_k^0)_+^*=(\Sigma_{k+1}^0)_+^*$. We denote the determinacy of games $G(A;\omega^{<\omega})$ for $A\in\Gamma$ by Det(Γ) [respectively, DetΓ]. By indiscernibles for an inner model M exist, we mean there exists a class C of indiscernibles for M, closed and unbounded beneath every uncountable cardinal. The following results are known:

Theorem 1. Let $\#^\infty$ denote the (partial) sharp function on all sets so that $\#^\infty(x)$ codes indiscernibles for $L(x)$. Let $\#_k$ be the restriction of $\#^\infty$ to objects of type k (so that $\#_1$ is the sharp function on the reals). Let $SH(\#)$ abbreviate $L[\#]\models$ "# is total."

(i) [Du2,4] If $k<\omega$ and $SH(\#_k)$, then Det$(\Pi_k^0)^*$ and Det$(\Sigma_{k+1}^0)^*$.

(ii) If $\omega\le k<\omega_1^{CK}$ and $SH(\#_k)$, then Det$(\Sigma_k^0)^*$ (so that Det$(\Pi_{k-1}^0)^*$ whenever k is a successor ordinal).

(iii) [Martin] If Det$(\Pi_1^1)^*$, then $SH(\#^\infty)$ and the converse fails to hold.

(iv) [Martin] If every real has a sharp and if Det$(\Pi_1^1)^*$, then indiscernibles for $L[\#^\infty]$ exist.

Theorem 2.

(i) [Du2,4] For $k < \omega$, indiscernibles for $L[\#_k]$ exist $\Leftrightarrow$ $\mathrm{Det}(\Pi^0_k)^*_+ \Leftrightarrow \mathrm{Det}(\Sigma^0_{k+1})^*_+$.

(ii) For $\omega \leq k < \omega_1^{CK}$, indiscernibles for $L[\#_k]$ exist $\Leftrightarrow$ $\mathrm{Det}(\Sigma^0_k)^*_+$ ($\Leftrightarrow$ $\mathrm{Det}(\Pi^0_{k-1})^*_+$ if k is a successor ordinal).

(iii) Indiscernibles for $L[\#^\infty]$ exist iff $\mathrm{Det}(\Pi^1_1)^*_+$.

Martin shows that $SH(\#_k)$ and $SH(\#^\infty)$ cannot be equivalent to the determinacy of a sufficiently absolute class (see the comments at the end of §1 of [Du4]) and therefore the converses of (i), (ii), and (iii) in Theorem 1 fail. By Theorems 1(iii) and 2(iii), $\mathrm{Det}(\Pi^1_1)^*$ is strictly stronger than $SH(\#^\infty)$ but no stronger than the existence of indiscernibles for $L[\#^\infty]$. With such existence of indiscernibles being only slightly stronger than $SH(\#^\infty)$, the following has been conjectured:

Conjecture. $\mathrm{Det}(\Pi^1_1)^* \Leftrightarrow \mathrm{Det}(\Pi^1_1)^*_+$ ($\Leftrightarrow$ by Theorem 2(iii), indiscernibles for $L[\#^\infty]$ exist).

Theorem 1(iv) seems to support this.

[Du4] proves (i) of Theorems 1 and 2, but relies on [Du3] which proves these results for $k = 1$. Martin and the author independently discovered the $\Rightarrow$ direction of Theorem 2(iii); this is the Main Theorem of this paper and its proof is given in §4. Martin and Welch proved the $\Leftarrow$ direction of Theorem 2(iii) around 1986; its proof is similar to that of the $\Leftarrow$ direction of Theorem 2(i,ii).

In §3 we generalize Theorem 2(iii). Fix $k < \omega_1^{CK}$. $\#^{(k+1)\infty}(x)$ is defined so that it codes indiscernibles for $L(x)[\vec{\#}^{k\infty}]$, where $\vec{\#}^{k\infty} = \langle \#^{(\beta+1)\infty} | \beta < k \rangle$. Let

($\bowtie$) $(x, n) \in (\langle B_i | i < k \rangle)$ iff for some $i < k$,

$B_i(x, n)$ and (i, n) is lexicographically least such that $B_i(x, n)$.

$(k * \Pi^1_1)$ is defined to be the collection of all $(\langle B_i | i < k \rangle)$ for which each $B_i \in \Pi^1_1$ and $\langle B_i | i < k \rangle$ is a Π^1_1 sequence. $(k * \Pi^1_1)^*_+ =_{df} ((k * \Pi^1_1))^*_+$. In §3, we show the $\Rightarrow$ direction of the following:

Theorem 3.5. Let $k < \omega_1^{CK}$. Indiscernibles exist for $L[\vec{\#}^{k\infty}]$ iff $\mathrm{Det}(k * \Pi^1_1)^*_+$.

Theorem 3.5 for $k = 0$ is a well-known result of Martin since $(\emptyset)^*_+ = \bigcup_{\beta<\omega^2} \beta - \Pi^1_1$ and $\vec{\#}^{0\infty} = \emptyset$.

Each of the proofs of the "getting determinacy" direction of Theorems 1, 2, and 3.5 combines some "form" of Borel Determinacy with the standard use of ordinal auxiliary moves. Martin orginally proved Borel Determinacy in [Ma2], but the proof was somewhat complicated by the use of a priority argument. Moschovakis later discovered a simpler proof for Finite Borel Determinacy (see Theorem 6F.1 on page 358 of [Mo]). In [Ma3], Martin introduced the concepts of covering and unraveling, and reorganized his original proof to provide a (simpler) purely inductive proof of Borel Determinacy. He later generalized this by proving quasi-Borel Determinacy in [Ma4]. (Martin's proof of Borel Determinacy in [Ma2,3] are for countable trees, whereas there is no such restriction for the quasi-Borel result; indeed, the Borel and quasi-Borel sets are the same for countable trees.)

To prove the "getting determinacy" direction of part (i) of Theorems 1 and 2 (where k is finite):

- one can get by with using the (simpler) Moschovakis' proof of Finite Borel Determinacy [Mo] and this is in fact what I use in [Du2,3,4,5].
- For $k = 1$, instead of Borel auxiliary moves, one can use Wolfe's proof (see Theorem 6A.3 on page 290 of [Mo]) of $\mathbf{\Sigma^0_2}$ determinacy (applied to Π^0_1 sets) and indeed my original proof did this.

To prove the "getting determinacy" direction of part (ii) of Theorems 1 and 2 (where $\omega \leq k < \omega_1^{CK}$), it is best to take advantage of Martin's methods in [Ma2] involving coverings and unraveling. We shall need to use quasi-Borel Determinacy for the results in this paper, i.e. for the "getting determinacy" direction of Theorems 2(iii) and 3.5.

By work of Donald A. Martin and Philip Welch [We], the determinacy strength of $\omega^2 - \Pi^1_1$ is known, both with respect to an indiscernibility property and with respect to mice. In 1985, Martin discovered an approximate characterization of $\mathrm{Det}(\omega^2 - \Pi^1_1)$ in terms of an indiscernibility property similar to (iii) below. In the 1990's, Welch found a mouse-equivalence for Det($\omega^2 - \boldsymbol{\Pi^1_1}$), and by 1994, proved the following:

Theorem 3. [We] Let $\alpha > 0$ be a recursive ordinal. The following are equivalent:

(i) Det($\omega^2\alpha - \Pi^1_1$).

(ii) There exists a clever α-mouse.

(iii) (Indiscernibility Property) There is a class C, closed and unbounded beneath every uncountable cardinal, so that for any two $\omega\alpha$ ascending sequences $\vec{c}$, $\vec{d}$ from C, setting $\mathcal{A}[\vec{c}] = \langle L_\gamma[\vec{c}], \in, \vec{c}\rangle$ equal to the smallest transitive admissible set containing the function $\vec{c}$, we have:

$$\mathcal{A}[\vec{c}] \models \sigma \Leftrightarrow \mathcal{A}[\vec{d}] \models \sigma$$

for any sentence σ of the form $\exists x_0 < c_{i_0} \exists x_1 < c_{i_1} \cdots \exists x_j < c_{i_j} \Psi$ where $j < \omega$ and Ψ is a boolean combination of Π_1 and Σ_1 formulae in the language $\mathcal{L}_{\{\dot{\in}, \dot{\vec{c}}\}}$.

Whether a mouse is clever (see (ii) above) depends on its "Q-structure." Theorem 3.10 (of this paper) shows that $\bigcup\limits_{k<\omega_1^{CK}} (k * \Pi^1_1)^*_+ \subseteq \Delta(\omega^2 - \Pi^1_1)$. From Welch's analysis of Q-structures [We], it is likely that $\bigcup\limits_{k<\omega_1^{CK}} (k * \Pi^1_1)^*_+$ lies near the bottom of $\Delta(\omega^2 - \Pi^1_1)$. It is clear how to characterize the determinacy of classes slightly above $\bigcup\limits_{k<\omega_1^{CK}} (k * \Pi^1_1)^*_+$ but we do not yet have characterizations of determinacy of some sufficiently absolute classes whose union equals $\Delta(\omega^2 - \Pi^1_1)$.

The weakest large cardinal property and smallest class considered so far in this Introduction are $SH(\#_1)$ and $(\Pi^0_1)^*$. Below these are (the existence of) $0^{k\#}$, where $0^{1\#} = 0^\#$ and $0^{(k+1)\#} = (0^{k\#})^\#$, and the classes $(k * \Sigma^0_1)^*$ and $(k * \Sigma^0_1)^*_+$ (these classes are defined analogous to that of $(k * \Pi^1_1)^*_+$ above). In [Du1], we show for $k < \omega$,

(a) $$0^{k\#} \Leftrightarrow \mathrm{Det}(k * \Sigma^0_1)^* \Leftrightarrow \mathrm{Det}((k-1) * \Sigma^0_1)^*_+.$$

It is well-known that:

(b) $$0^\# \text{ exists iff } \mathrm{Det}(\beta - \Pi^1_1) \text{ for some (all) } \beta < \omega^2.$$

Friedman [Fr] showed that the existence of $0^\#$ implies the determinacy of $3 - \Pi^1_1$ games. Then Martin showed that all $\bigcup\limits_{\beta<\omega^2} (\beta - \Pi^1_1)$ games are determined iff $0^\#$ exists. In 1975, Martin showed that $\mathrm{Det}\big(3-\Pi^1_1\big)$ implies $0^\#$ exists; soon after, Harrington [Ha] showed that the determinacy of Π^1_1 games implies the existence of $0^\#$. (b) thus follows.

Between each $(k*\Pi^1_1)^*_+$ and $\big((k+1)*\Pi^1_1\big)^*$ is a rich hierarchy of classes. By combining the proofs of Theorems 1(i),(ii) and 2, one obtains level-by-level results generalizing Theorems 1(i,ii) and 2. We describe the simplest such case, generalizing the results in [Du3,5]. For simplicity, we only consider finite k, β, and γ. Inductively define the partial sharp function $\beta\#^{k\infty,1}_\gamma$ on the reals so that:

- $\#^{k\infty,1}_{\gamma+1}(r)$ codes indiscernibles for $L(r)[\vec{\#}^{k\infty,1}_\gamma]$, where $\vec{\#}^{k\infty,1}_\gamma$ is the concantenation of $\vec{\#}^{k\infty} = \langle \#^{(\beta+1)\infty} | \beta < k\rangle$ followed by $\langle \#^{k\infty,1}_{\xi+1} | \xi < \gamma\rangle$.
- $(\beta+1)\#^{k\infty,1}_\gamma(r) =_{df} \#^{k\infty,1}_\gamma(\beta\#^{k\infty,1}_\gamma(r))$ and $0\#^{k\infty,1}_\gamma(r) =_{df} r$.

$(k*\Pi^1_1, \gamma*\Pi^0_1, \beta*\Sigma^0_1)$ is defined analogous to that of $(k*\Pi^1_1)$ above: it is the collection of all $(\langle B_i | i < k+\gamma+\beta\rangle)$ (see $(\bowtie)$) for which $\langle B_i | i < k+\gamma+\beta\rangle$ is a Π^1_1 sequence and

$$B_i \in \begin{cases} \Pi^1_1 & \text{if } 0 \le i < k, \\ \Pi^0_1 & \text{if } k \le i < k+\gamma, \\ \Sigma^0_1 & \text{if } k+\gamma \le i < k+\gamma+\beta. \end{cases}$$

Theorem 4.

(i) If $\beta\#^{k\infty,1}_{\gamma+1}(0)$ exists and $L(\beta\#^{k\infty,1}_{\gamma+1}(0))[\vec{\#}^{k\infty,1}_\gamma] \models$ "$\#^{k\infty,1}_\gamma$ is total," then $\mathrm{Det}(k*\Pi^1_1, \gamma*\Pi^0_1, \beta*\Sigma^0_1)^*$.

(ii) The existence of $(\beta+1)\#^{k\infty,1}_{\gamma+1}(0)$ is equivalent to $\mathrm{Det}(k*\Pi^1_1, \gamma*\Pi^0_1, \beta*\Sigma^0_1)^*_+$.

The proof of Theorem 4 for the case $k=0$ is given in [Du3,5], with [Du3] providing the "using determinacy" direction and [Du5] providing the "getting determinacy" direction.

If $f : \omega_1^{CK} \to \omega_1^{CK}$, $\omega_1^{CK} > \alpha(m) > \alpha(m-1) > \cdots > \alpha(\ell) \ge \omega > \alpha(1) \ge 1$ is a descending sequence of ordinals, and if $\{i | f(i) \ne 0 \ \wedge\ 0 < i < \omega_1^{CK}\} \subseteq \{\alpha(j) | j < m\}$, then

$$\begin{aligned} \Pi(f) =_{df} \Big(& f(\omega_1^{CK}) * \Pi^1_1, f(\alpha(m)) * \Sigma^0_{\alpha(m)}, f(\alpha(m-1)) * \Sigma^0_{\alpha(m-1)}, \ldots, f(\alpha(\ell)) * \Sigma^0_{\alpha(\ell)}, \\ & f(\alpha(\ell-1)) * \Pi^0_{\alpha(\ell-1)}, \ldots, f(\alpha(1)) * \Pi^0_{\alpha(1)}, f(0) * \Sigma^0_1 \Big). \end{aligned}$$

We are mainly interested in considering classes of the form $(\Pi(f))^*$ and $(\Pi(f))^*_+$ since $(\Pi^0_n)^* = (\Sigma^0_{n+1})^*$, $(\Pi^0_n)^*_+ = (\Sigma^0_{n+1})^*_+$, and since $(\Gamma_1, \Gamma_2) = (\Gamma_2)$ when Γ_1 and its dual are subclasses of Γ_2.

Let $t(f)$ be the least $\xi > 0$ such that $f(\xi) > 0$. One can inductively define partial sharp functions $\#_f$ on objects of type $t(f)$ so that, letting $1\#_{f+1}(r)$ code indiscernibles for $L(r)[\vec{\#}_f]$ and setting $(\beta+1)\#_{f+1}(r) =_{df} \#_{f+1}(\beta\#_{f+1}(r))$, we have the natural generalizations of Theorems 1(i,ii), 2, 3.5, and 4:

Theorem 5.

(i) If $t(f) < \omega_1^{CK}$ and $L(f(0)\#_{f+1}(0))[\vec{\#}_f] \models$ "$\#_f$ is total," then $\mathrm{Det}(\Pi(f))^*$.

(ii) $(f(0)+1)\#_{f+1}(0)$ exists (i.e. $L(f(0)\#_{f+1}(0))[\vec{\#}_f]$ has indiscernibles) iff $\mathrm{Det}(\Pi(f))^*_+$.

In our current draft of [Du6], we prove Theorem 5 for the special case in which the only values on which f is nonzero are finite. Possibly [Du6] will be rewritten to prove the more general result.

In Section 1 of this paper, we examine the classes $(\Pi^1_1)^*$ and $(\Pi^1_1)^*_+$; in particular, we review the definitions of $\beta - \Pi^1_1$, $(\Gamma)^*$, and $(\Gamma)^*_+$, we prove the Normal Form Lemma for $(\Pi^1_1)^*_+$ (used in the proof of Theorem 1(iii)), and we show $(\Pi^1_1)^*_+$ properly contains $(\Pi^1_1)^*$ but is contained in $\Delta(\omega^2 - \Pi^1_1)$. Section 2 is devoted to proving the "getting determinacy" direction of Theorem 2.3(iii); this is actually proven in subsection 2.3 (see Theorem 2.9). In subsection 2.1, we state the version of quasi-Borel determinacy which we shall need to prove Theorem 2(iii) and review the necessary definitions. In subsection 2.2, we review the standard introduction of ordinal auxiliary moves and $(\Sigma^0_1)^*$ determinacy, so that we can avoid these (usual) details in the proof of Theorem 2(iii). Section 3 is devoted to generalizing Theorem 2.3(iii) by proving Theorem 3.5 and to examining $(k * \Pi^1_1)^*_+$.

Martin's upcoming book [Ma5] on determinacy covers all the relevant material to this paper and [Du1,2,3,4,5,6]. Borel determinacy, quasi-Borel determinacy, and the standard use of ordinal auxiliary moves are covered in [Ma5]. To varying degrees, the proofs of Theorems 1, 2, 3.5, 4, (a), (b), and 5 are given, outlined, or provided as exercises in [Ma5] (especially see Chapter 5).