Section 2.

In subsection 2.3, we prove the Main Theorem 2.9 of this paper. In subsection 2.1 we provide the necessary material concerning quasi-Borel determinacy. In subsection 2.2, we review the standard use of ordinal auxiliary moves.

Section 2.1 Quasi-Borel determinacy.

[Ma4] is the reference for the material and notation in this subsection. Given a tree T, $[T]$ denotes the set of plays from T and $G(E,T)$ denotes the game played on T with payoff set $E \cap [T]$ (so $G(E,T)$ and $G(E \cap [T],T)$ are the same game). If $p \in T$, then T_p denotes the tree of positions in T which are compatible with p. We denote the strategies on T by $\mathcal{S}(T)$.

Definition 2.1. Let T be a tree and $A \subseteq [T]$. $A \in \mathbf{\Sigma^1_1}$ iff there is a closed $C \subseteq [T] \times \omega^\omega$ such that

$$(\forall x \in [T])(x \in A \leftrightarrow (\exists y \in^\omega \omega)\langle x, y\rangle \in C).$$

$A \in \mathbf{\Pi^1_1}$ iff $[T] \setminus A \in \mathbf{\Sigma^1_1}$. $\mathbf{\Delta^1_1} = \mathbf{\Sigma^1_1} \cap \mathbf{\Pi^1_1}$. △

Martin [Ma4] proved for every tree T, $\mathbf{\Delta^1_1}$ coincides with the class of quasi-Borel sets and quasi-Borel games are determined. We review some of the basic notions from [Ma4].

Definition 2.2. Let M be an inner model, $T \in M$ a tree, and $k < \omega$. An *M-k-covering* of T is a triple $(\tilde{T}, \pi, \phi) \in M$ where

(a) $\tilde{T}$ is a tree;

(b) $\pi : \tilde{T} \to T$;

(c) $\tilde{t} \subseteq \tilde{u} \to \pi(\tilde{t}) \subseteq \pi(\tilde{u})$, for all $\tilde{t}$ and $\tilde{u}$ belonging to $\tilde{T}$;

(d) length$(\pi(\tilde{t}))$=length$(\tilde{t})$ for all $\tilde{t} \in \tilde{T}$;

(e) $\tilde{T}$ and T have the same members of length $\leq k$, and π is the identity on $\{\in \tilde{T} | \text{length}(t) \leq k\}$;

(f) $\phi : \mathcal{S}(\tilde{T}) \cap M \to \mathcal{S}(T)$ and each $\phi(\tilde{\sigma})$ is a strategy for the same player as is $\tilde{\sigma}$;

(g) for all $n < \omega$, $\phi(\tilde{\sigma})$ restricted to positions of length $\leq n$ depends only on $\tilde{\sigma}$ restricted to positions of length $\leq n$;

(h) (lifting property) if x is a play consistent with $\phi(\tilde{\sigma})$, then there is a play $\tilde{x}$ consistent with $\tilde{\sigma}$ such that $\pi(\tilde{x}) = x$. △

A $V - k$-covering is simply a k-covering as defined on page 283 of [Ma4]. An $M - k$-covering is not necessarily a k-covering since ϕ only acts on strategies in M.

Definition 2.3. A $M - k$-covering $(\tilde{T}, \pi, \phi)$ *unravels* a set $A \subseteq [T]$ if $\pi^{-1}(A)$ is clopen. △

Theorem 2.4. [Ma4] Let T be a tree and E be a quasi-Borel subset of $[T]$. Then for each $k < \omega$ there is a k-covering of T which unravels E and $G(E, T)$ is determined. ∎

We need following variant of the Theorem 2.4.

Lemma 2.5. (Martin) Let M be an inner model and $k < \omega$. Let $S \in M$ be a tree and E a quasi-Borel subset of $[S] \times \omega$ with code in M. Then there is $M - k$-covering (T, π, ϕ) of S which unravels E.

Proof: One constructs the $M-k$-covering by running in M the proof of Theorem 2 of "An Extension of Borel deeterminacy," and proving by induction on the quasi-Borel rank of E (with respect to the given code) that the resulting M-covering has the required properties. The induction steps are obvious. The coverings are built by composition at successor and inverse limits are limits.) And the case E is open or closed is clear from inspection of that part of the Borel determinacy proof. ∎(2.5)

Martin proved the above variant of quasi-Borel determinacy so that one can prove the Main Theorem 2.9 by mainly working in V.

Section 2.2. Indiscernibles and auxiliary ordinal moves.

Theorem 2.6. [Ma5] Let T be a tree and assume $\#(T)$ exists. If E is a $\beta - \mathbf{\Pi^1_1}$ subset of $[T]$ for some $\beta < \omega^2$, then $G(E, T)$ has a w.s. $s \in L(\#(T))$.

Theorem 2.6 is Martin's Theorem, is well-known, and is proven in Chapter 5 of [Ma5]. In its proof, an open auxiliary game G' corresponding to $G(E, T)$ is defined. G' has ordinal auxiliary moves ξ_i $(i < \omega)$. Using indiscernibles for $L(T)$, a w.s. for $G(E, T)$ is constructed by integrating a w.s. for G' with respect to these ordinal auxiliary moves. We will need to similarly introduce such ξ_i in the proof of the Main Theorem 2.9. This set-up is well-known. We review it here and then refer to this set-up in the proof of the Main Theorem 2.9 (but skip the proof of Theorem 2.6). (See Theorem 0.8 of [Du1] for complete proof and description of G' for the case in which $T = \omega^{<\omega}$.)

Definition 2.7. (Description of G'). Suppose E is $(\omega \cdot m + 1) - \Pi^1_1$. (We include the $+1$ here only to match the situation of the Main Theorem 2.9.) In G' an ordinal auxiliary move ξ_i is played with each integer move $x(i)$. Each ξ_i is thought of as some particular ξ^α_j ($j < \omega$, $\alpha \leq \omega \cdot m$) by setting $\xi^\alpha_j = \xi_{\pi(\alpha,j)}$ where $\pi : (\omega \cdot m + 1) \times \omega \to \omega$ is a fixed recursive bijection satisfying:

- ξ^α_j is played before ξ^α_{j+1}, i.e. $\pi(\alpha, j) < \pi(\alpha, j+1)$.
- $\xi^{\omega \cdot n+k}_0$ is played before $\xi^{\omega \cdot n+k+1}_0$, i.e. $\pi(\omega \cdot n + k, 0) < \pi(\omega \cdot n + k + 1, 0)$.
- ξ^α_0 is played by I $\Leftrightarrow$ α is even, i.e. $\pi(\alpha, j)$ is even $\Leftrightarrow$ α is even.

Let $\langle D_\alpha | \alpha \leq \omega \cdot m \rangle$ witness that E is $((\omega \cdot m) + 1) - \Pi^1_1$. A well-known result of Kleene's [Kl] is that for $\alpha \leq (\omega \cdot m)$ and x a real, there exist recursive orderings $\prec^\alpha_{\bar{x}(i)}$ such that $\prec^\alpha_{\bar{x}(i)}$ is a linear ordering of $0, 1, 2, ..., i$ and is a subordering of $\prec^\alpha_{\bar{x}(i+1)}$ and

$$(\$) \qquad x \in A_\alpha \text{ iff } \prec^\alpha_x =_{df} \bigcup_{i \in \omega} \prec^\alpha_{\bar{x}(i)} \text{ is a wellordering.}$$

(See Lemma 2.4 [Du4] for more detail.)

If the map $j \mapsto \xi^\alpha_j$ from ω under the ordering $\prec^\alpha_x$ and into the ordinals is order-preserving, then $\prec^\alpha_x$ is a wellordering and by (\$), $x \in A_\alpha$. When each such map is order-preserving, we say the (corresponding) play is well-played; otherwise, it is badly played. Similarly we define well-played and badly played for positions. If a play is badly played, then there is a position of a least length i which is badly played and we naturally say I [resp. II] played badly if i is odd [resp. even]. ■(2.6)

One can get slightly more determinacy than what is stated in Lemma 2.6 (by essentially the same proof).

Lemma 2.8. [Du1] Let T be a tree and assume $\#(T)$ exists. If E is a $(\mathbf{\Sigma^0_1})^*$ subset of $[T]$, then $G(E, T)$ has a w.s. $s \in L(\#(T))$.

Proof: This is proven in Section 1 of [Du1] for the case in which $T = \omega^{<\omega}$. We briefly outline the proof. Let $B \in \mathbf{\Sigma^0_1}$ and $\vec{A}$ witness that $E = B^*(\vec{A})$ is a $(\mathbf{\Sigma^0_1})^*$ subset of $[T]$. Recall $\mathcal{D}(\vec{A}_{\omega\cdot i})$ is the $\omega \cdot i - \Pi^1_1$ set witnessed by $\langle A_\alpha | \alpha < \omega \cdot i\rangle$. Let $B_i = \{x | B(x,i)\}$. Let a position p in T have ordinal 0 iff: $[T_p] \subseteq B_i$ for some $i < \omega$, and for the least such i, player I has a w.s. for $G(\mathcal{D}(\vec{A}_{\omega\cdot i}), T_p)$. Otherwise inductively define the ordinal rank of a position as usual:

- A position p of odd length has ordinal α iff every position p' extending p by one (exactly) move has ordinal rank $\leq \alpha$.
- A position p of even length has ordinal α iff there exists a position p' with ordinal rank $< \alpha$ and extending p by one move.

By Lemma 2.6, $G(\mathcal{D}(\vec{A}_{\omega\cdot i}), T_p)$ has a w.s. $s^p_i \in L(\#(T))$. Note that:

(a) If p has rank 0, then s^p_i must be a w.s. for player I where i is least such that $[T_p] \subseteq B_i$.

(b) If p has no rank and $[T_p] \subseteq B_i$ for some least $i < \omega$, then s^p_i must be a w.s. for player II.

Suppose $\langle\ \rangle$ has ordinal rank. We implicitly define a w.s. $\sigma \in L(\#(T))$ as follows: I first plays to reduce ordinal rank (see Theorem ? of [Du4]), until a position p of rank 0 is reached and then for appropriate i, I follows the w.s. s^p_i by (a). Suppose x is a resulting play (consistent with σ). Since $p \subseteq x$ has ordinal 0, there exists i such that $x \in B_i$ and x is consistent with s^p_i so that $x \in \mathcal{D}(\vec{A}_{\omega\cdot i})$. Thus $x \in E$.

Now suppose $\langle\ \rangle$ has no ordinal rank. We implicitly define a w.s. $\tau \in L(\#(T))$ as follows: II plays positions with no ordinal rank (see Theorem ? of [Du4]) until (if ever) a position p is reached such that for some least $i < \omega$ $[T_p] \subseteq B_i$, in which case he follows the w.s. s^p_i given by (b). Suppose x is a resulting play (consistent with τ). If a position p was reached such that $T_p \subseteq B_i$ (for some least $i < \omega$), then $x \in B_i$ and since x is consistent with s^p_i, $x \notin \mathcal{D}(\vec{A}_{\omega\cdot i})$. If no such p was reached, then $x \notin \bigcup_{i\in\omega} B_i$ so that $\forall i \neg B(x,i)$. Thus, in both cases, $x \notin E$ and x is a win for II. ∎(2.8)

Remark. By Lemma 1.3 [Du1], one can select $\vec{A}$ in the proof of Theorem 2.8 to also satisfy:

- if $B(x,i)$ and $B(x,j)$, then $x \in \mathcal{D}(\vec{A}_{\omega\cdot i})$ iff $x \in \mathcal{D}(\vec{A}_{\omega\cdot j})$.

Thus, with such selection, one need not worry about i being least in the proof of Theorem 2.8.

Section 2.3. The Main Theorem.

Recall from Theorem 1(iii) of the Introduction that $(\Pi^1_1)^*$ determinacy is stronger than $SH(\#^\infty)$. We know show that a slightly stronger large cardinal property than $SH(\#^\infty)$ implies the determinacy of not only $(\Pi^1_1)^*$ but in fact the slightly larger class $(\Pi^1_1)^*_+$.

Theorem 2.9. If $L[\#^\infty]$ has indiscernibles, then $\mathrm{Det}(\Pi^1_1)^*_+$.

Proof: For the remainder of this section, fix $A \in (\Pi^1_1)^*_+$ and by the Normal Form Lemma, let $B \in \Pi^1_1$, $\vec{E} = \langle E_\alpha \mid \alpha < \omega^2\rangle$, and $\vec{D} = \langle D_\alpha \mid \alpha \le \omega\cdot m\rangle$ strongly witness that $A = B^*(\vec{E},\vec{D}) \in (\Pi^1_1)^*_+$.

First introduce ordinal auxiliary moves ξ_i exactly as in Definition 2.7. Let T be the tree such that an ordinal auxiliary move ξ_i is played with each integer move $x(i)$. As in Definition 2.7, each ξ_i is some ξ^α_j $(j < \omega, \alpha \le \omega \cdot m)$ and

(i) if ξ^α_j for $j < \omega$ are played well, then $x \in D_\alpha$.

If $p \in T$ and $y \in [T]$, obtain $f(p)$ and $f(y)$ by deleting the ordinal auxiliary moves from p and y. Player I wins $G(A_T, T) \Leftrightarrow$ either II badly loses or (noone badly loses and $f(y) \in A$).

By the proof of Lemma 2.6, we have:

Lemma 2.10. If $G(A_T,T)$ has a w.s. $s \in L[\#^\infty]$, then the game A is determined.

Proof of Lemma: In the proof of Lemma 2.6, indiscernibles for L are used to obtain $\beta - \Pi^1_1$ determinacy for $\beta < \omega^2$. Analogously, use indiscernibles for $L[\#^\infty]$ to integrate $s \in L[\#^\infty]$ with respect to the ordinal auxiliary moves ξ_i and obtain a w.s. for the game A. ∎(2.10)

Let $T' =_{df} \{p \in T | \exists y \in [T_p]$ such that for each α, the ordinal moves ξ^α_j of y are played well (with respect to D_α)$\}$. $[T']$ is a $\mathbf{\Pi^0_1}$ subset of $[T]$. Let

$$B_i =_{df} \{y \in [T'] | B(f(y), i)\},$$

a $\mathbf{\Pi^1_1}$ subset of $[T]$. For $y \in [T']$, $\exists n B(f(y), n)$ by (i) so that, since B is uniformized, $\exists! n B(x, n)$ and $(y \notin B_i \Rightarrow \exists n \neq i B(x, n))$. Hence for $y \in [T]$:

$$\neg B(f(y), i) \Leftrightarrow \text{ either } y \notin [T'] \text{ or } (y \in [T'] \text{ and } \exists n \neq i B(f(y), n))\,.$$

Therefore each B_i is a $\mathbf{\Delta^1_1}$ subset of $[T]$ with code in $L[\#^\infty]$. By Martin's refinement of quasi-Borel determinacy (Lemma 2.5), there exists a $L[\#^\infty]$-covering $\left(\tilde{T}, \pi, \phi\right) \in L[\#^\infty]$ of T which unravels each B_i (so each $\pi^{-1}(B_i)$ is clopen). (Recall $\left(\tilde{T}, \pi, \phi\right)$ has all the properties of a covering of T except ϕ only acts on strategies in $L[\#^\infty]$–we will need this to assure that $\phi(s)$ (defined below) is an element of $L[\#^\infty]$, which in turn is necessary to integrate $\phi(s)$ with respect to the ξ_i's.)

Now we compute $\pi^{-1}(A_T)$:

$z \in \pi^{-1}(A_T)$ iff either $\pi(z)$ is badly lost for II, or

$\pi(z)$ is not badly lost and $\exists i[z \in \pi^{-1}(B_i)\ \&\ f(\pi(z)) \in \mathcal{D}(\vec{A}_{\omega \cdot i})]$.

Since $\pi^{-1}(B_i)$ is clopen, we have $\pi^{-1}(A_T)$ is $(\mathbf{\Sigma^0_1})^*$ with code in $L[\#^\infty]$ so that by Lemma 2.8, we have

$$G(\pi^{-1}(A_T), \tilde{T}) \text{ has a w.s. } s \in L[\#^\infty].$$

Hence, $\phi(s) \in L[\#^\infty]$ is a w.s. for $G(A_T, T)$. By Lemma 2.10, $G(A)$ has a w.s. in $L(\#^{2\infty}(0))$. ∎(2.9)

Remark 1. All the codes mentioned are in L–possibly this (or something similar) should provide the lightface notions (of Δ^1_1, etc.). Also, in the calculation of $\pi^{-1}(A_T)$, one might put $\exists i$ on the "outside" and let the $\mathbf{\Sigma^0_1}$ set be the collection of all $z \in [\tilde{T}]$ such that either $\pi(z)$ is badly loss for II or $z \in \pi^{-1}(B_i)$.

Remark 2. Martin's proof of the above theorem is slightly different. He uses absoluteness to work in $L[\#^\infty]$ and then uses quasi-Borel determinacy within $L[\#^\infty]$.

D. A. Martin and Philip Welch proved the converse to Theorem 2.9 around 1986, so we have:

Theorem 2.11. $L[\#^\infty]$ has indiscernibles iff $\mathrm{Det}(\Pi^1_1)^*_+$.

The proof of the converse of Theorem 2.9 is similar to that of the using determinacy direction of Theorem 2(i) (which is proved in detail in Section 2 of [Du4] and for $k = 1$ in [Du2]). However, in [Du4], when providing the usual Σ^1_1-boundedness argument, I include $B(x, n)$ in my definition of the Σ^1_1 set, which is fine there since $B \in \Pi^0_k$ in [Du2,4]. The Σ^1_1-boundedness argument works without including B in the definition of the Σ^1_1 set, which is important here since $B \in \Pi^1_1$.

For the remainder of this section, we briefly describe a $(\Pi^1_1)^*_+$ check-up game $B^*(\vec{A}, \vec{D})$ but only for relevant plays (plays which end up being considered in the proof to converse of Theorem 2.9). Providing this description and Section 2 of [Du2,4], we leave for the reader to define the actual check-up game and prove of the converse of Theorem 2.9.

Fix a formula $\varphi = \varphi(v_0, v_1, ..., v_{\ell-1})$. As usual, producing indiscernibles for $L[\#^\infty]$ reduces to showing that there is a club subset C of ω_1 such that for any $\rho_0 < \rho_1 < \cdots < \rho_{\ell-1}$ from C, $L_{\rho_{\ell-1}}[\#^\infty] \models \varphi_\lambda[\#^\infty, \rho_0, \rho_1, ..., \rho_{\ell-2}]$, where $\varphi_\lambda =_{df} \begin{cases} \varphi & \text{if } \lambda = I, \\ \neg\varphi & \text{if } \lambda = II \end{cases}$. We think of player λ playing an ω-model M^λ and ω^2 (possible) orderings $\prec^\lambda_i$ for $i < \omega^2$. When $\prec^\lambda_i$ is a wellordering for $\lambda = I, II$ and $i < \omega \cdot j$, we define (from these) the ordinal ρ_{j-1} in the usual way. $B(x, n)$ is meant to hold when the players' ω-models disagree about a sharp. If $\forall n \neg B(x, n)$, then we decide who wins by checking the $\prec^\lambda_i$ for $i < \omega \cdot \ell$, and if each of these is a wellordering, then we decide who wins by checking that M^λ is an ω-model of order type $\rho_{\ell-1}$ satisfying a sufficient amount of ZFC, $V = L[\#^\infty]$, and $\varphi_\lambda[\#^\infty, \rho_0, \rho_1, ..., \rho_{\ell-2}]$. If $\exists n B(x, n)$ holds, then for simplicity we can check the same exact items as we have described in the case $\forall n \neg B(x, n)$.

Define $B \in \Pi^1_1$ so that if $B(x, n)$ holds, then (for relevant plays) the two ω-models disagree about the of sharp of some set y in their common wellfounded part with respect to some formula $\psi = \psi(v_0, v_1, ..., v_{m-1})$. In this case, require that at least one of the models believes $y^\#$ exists and contains the Gödel number of ψ as an element; the other model either believes $y^\#$ does not exist or $\neg\psi$ has Gödel number in $y^\#$. Also in this case, require $n \geq m + \ell$.

If $B(x, n)$ and both players satisfy all the items required when $\forall n \neg B(x, n)$, then we decide who wins by checking whether the orderings $\prec_i^\lambda$ for $\omega \cdot \ell \leq i < \omega \cdot m$ are wellorderings, and if all of these are, we use the ordinals $\rho_\ell, \rho_{\ell+1}, ..., \rho_{\ell+m}$ to check which player is correct concerning ψ (for instance, if M^I believes ψ is in the sharp of y, then I needs $L_{\rho_m} \models \psi[\rho_\ell, \rho_{\ell+1}, \rho_{\ell+2}, ..., \rho_{\ell+m-1}]$).