Section 3. Generalization of the Main Theorem.

In this section, we generalize the Main Theorem 2.9 of this paper by proving:

Theorem 3.5. If $k < \omega_1^{CK}$ and $L\left[\vec{\#}^{k\infty}\right]$ has indiscernibles, then $\mathrm{Det}\big(k * \Pi_1^1\big)_+^*$.

The proof is similar to that of the Main Theorem 2.9. First we need to define the class $\big(k * \Pi_1^1\big)_+^*$ and prove the corresponding Normal Form Lemma.

Definition 3.1. $(k * \Pi_1^1)$. Let $k < \omega_1^{CK}$ and let $\vec{B} = \langle B_i | i < k\rangle$ be such that each $B_i \subseteq (\omega^\omega) \times \omega$. $(\vec{B})(x, n)$ iff there exists i such that (i, n) is lexicographically least such that $B_i(x, n)$. If each $B_i \in \Gamma$, then $(\vec{B}) \in (k * \Gamma)$. If k is a successor ordinal, we sometimes write $(B_0, B_1, ..., B_{k-1})$ for $(\vec{B})$. If $C \in (n * \Pi_1^1)$ and $B \in (k * \Pi_1^1)$, we let (C, B) denote the obvious $((n + k) * \Pi_1^1)$ set.

The definitions of $(k * \Pi_1^1)^*$ and $(k * \Pi_1^1)_+^*$ are clear from Definitions 1.4, 1.5, and 3.1. $\vec{B} = \langle B_i | i < k\rangle$, $\vec{A}$, and $\vec{D}$ *witnesses* that $A \in (k * \Pi_1^1)_+^*$ has the obvious meaning (as does witnessing a set being $(k * \Pi_1^1)^*$).

In §2.3, we showed Theorem 3.5 for the special case $k = 1$. That proof used that "wlog the B is uniformized." Recall $B \in \Pi_1^1$ is *uniformized* if $\forall x$ $\exists$ at most one n such that $B\,(x, n)$. To prove Theorem 3.5 for the successor case, we first show that "wlog the last B of a $\big(k * \Pi_1^1\big)_+^*$ set is uniformized."

Lemma 3.2. (Normal Form Lemma for $((k + 1) * \Pi_1^1)^*$) If $A \in \big((k + 1) * \Pi_1^1\big)^*$, then there exist $C \in (k * \Pi_1^1)$, uniformized $B \in \Pi_1^1$, and $\vec{E}$ such that $A = (C, B)^*\left(\vec{E}\right)$ and $(\exists n C(x, n) \Rightarrow \exists n B(x, n)$.

Proof. Let $C \in (k * \Pi_1^1)$, $B \in \Pi_1^1$, and $\vec{A}$ witness that $A = (C, B)^*\left(\vec{A}\right) \in \big((k + 1) * \Pi_1^1\big)^*$. Let $C = (\langle C_i | i < k\rangle)$, where each $C_i \in \Pi_1^1$. Wlog $C \subseteq \bigcup_{i<k} C_i \subseteq B$, since

$$(C, B)^*\left(\vec{A}\right) = (C, C \cup B)^*\left(\vec{A}\right) = \left(C, (\bigcup_{i<k} C_i) \cup B\right)^*\left(\vec{A}\right)$$

and $(\bigcup_{i<k} C_i) \cup B \in \Pi_1^1$. Let $\hat{B}$ uniformize $B = (\bigcup_{i<k} C_i) \cup B$ so that

(i) $\forall x$ $\exists$ at most one n such that $\hat{B}\,(x, n)$,

(ii) $\hat{B}\,(x, n) \Rightarrow B\,(x, n)$, and

(iii) $\exists n B\,(x,n) \Leftrightarrow \exists n \hat{B}\,(x,n)$.

The idea is to stack the A_α's first with respect to $\hat{B}$ and then with respect to C. Let

$$x \in E_{\omega\cdot n+i} \Leftrightarrow \text{ either } \left[\exists m > n\hat{B}\,(x,m) \text{ and } (x \in A_{\omega\cdot n+i} \text{ or } \exists j \le nB\,(x,j))\right]$$

$$\text{or } \left[\exists m \le n\left(\hat{B}\,(x,m) \wedge x \in A_{\omega\cdot(n-m)+i}\right)\right].$$

Let $C'\,(x,n) \Leftrightarrow \exists m \le n[\hat{B}(x,m) \;\wedge\; C\,(x,n-m)]$. Then $(C,B)^*\left(\vec{A}\right) = \left(C',\hat{B}\right)^*\left(\vec{E}\right)$.

In our construction of $\left(C',\hat{B}\right)^*\left(\vec{E}\right)$, first we check $x \in A_\alpha$ until we reach m such that $\hat{B}(x,m)$. If we never reach such an m, then both $\forall n \neg C(x,n)$ and $\forall n \neg B(x,n)$ so that the A_α's are insignificant.

As "α increases through stages $\le m$ such that $\hat{B}(x,m)$," if we find a $j < m$ such that $B(x,j)$, then we make sure x doesn't fall out of $\vec{E}$ before stage m by adding the clause:

"$\exists j \le n\; B(x,j)$ and $\exists m > n\hat{B}(x,m)$."

Once we reach m such that $\hat{B}(x,m)$, we start checking for $\hat{n}$ such that $C(x,\hat{n})$. If no such $\hat{n}$ exists, then we are done. If such $\hat{n}$ exists, we start over in checking $x \in A_\alpha$, until we reach n such that $C(x,n-m)$. ■

Lemma 3.3. (Normal Form Lemma for $((k+1)*\Pi^1_1)^*_+$) If $A \in \left((k+1)*\Pi^1_1\right)^*_+$, then (for some $m < \omega$) there exist $C \in (k*\Pi^1_1)$, uniformized $B \in \Pi^1_1$, $\vec{A}$, and $\vec{D} = \langle D_\alpha | \alpha \le \omega\cdot m\rangle$ such that $A = (C,B)^*\left(\vec{A},\vec{D}\right)$, $(\exists n C(x,n) \Rightarrow \exists n B(x,n))$, and

$$\exists n B(x,n) \Leftrightarrow \forall \alpha \le \omega\cdot m(x \in D_\alpha).$$

Proof: Let $\hat{C} \in (k*\Pi^1_1)$, $\hat{B} \in \Pi^1_1$, $\vec{E}$, and $D \in \omega\cdot m - \Pi^1_1$ witness that A is $\left((k+1)*\Pi^1_1\right)^*_+$. By Lemma 3.2, there exist $C \in (k*\Pi^1_1)$, $B \in \Pi^1_1$, and $\vec{A}$ which witness that $A = (\hat{C},\hat{B})^*(\vec{E})$ is $((k+1)*\Pi^1_1)^*$ and such that $C \subseteq B$ and B is uniformized.

Let $\langle d_\alpha | \alpha < \omega\cdot m\rangle$ witness that $D \in \omega\cdot m - \Pi^1_1$. For $\alpha < \omega\cdot m$, let

$$x \in D_\alpha \Leftrightarrow x \in d_\alpha \text{ or } \exists n B(x,n).$$

Then C, B, $\vec{A}$, and $\vec{D} = \langle D_\alpha | \alpha \le \omega\cdot m\rangle$ are as stated in the lemma. ■

Definition 3.4. Recall $\vec{\#}^{k\infty} =_{df} \langle \#^{(\beta+1)\infty} | \beta < k\rangle$. For $k < \omega_1^{CK}$, $\#^{(k+1)\infty}$ is inductively defined to be a partial sharp function on all sets such that for $x \in dom(\#^{(k+1)\infty})$,

$\#^{(k+1)\infty}(x)$ is defined to be a real that codes indiscernibles for $L(x)[\vec{\#}^{k\infty}]$, where $L(x)[\vec{\#}^{k\infty}]$ is the least inner model containing the transitive closure of x and defined using relative constructability from $\vec{\#}^{k\infty}$. For more details concerning the definition of $L(x)[\vec{\#}^{k\infty}]$, see Definitions 1.4 [Du3] and 1.1 [Du4].

We now generalize Theorem 2.9 (which stated that if $L\left[\#^{\infty}\right]$ has indiscernibles, then $\text{Det}\left(\Pi_1^1\right)_+^*$).

Theorem 3.5. If $k < \omega$ and $L\left[\vec{\#}^{k\infty}\right]$ has indiscernibles, then every $\left(k * \Pi_1^1\right)_+^*$ game has a w.s. in $L(\#^{(k+1)\infty}(0))$.

Proof: The proof is by induction on k and Theorem 2.9 is the base step. This proof of the induction step is similar to that of Theorem 2.9, with our induction hypotheses replacing our use of Theorem 2.6. We outline here the changes to that proof.

Let $A \in \left(k * \Pi_1^1\right)_+^*$. By Lemma 3.3, there exist $C \in \left((k-1) * \Pi_1^1\right)$, uniformized $B \in \Pi_1^1$, $\vec{A}$, and $\vec{D} = \langle D_\alpha | \alpha \leq \omega \cdot m \rangle$ such that:

(a) $A = (C, B)^* \left(\vec{A}, \vec{D}\right)$.

(b) $(\exists n C(x,n) \Rightarrow \exists n B(x,n))$ and $[\exists n B(x,n) \Leftrightarrow \forall \alpha \leq \omega \cdot m \, (x \in D_\alpha)]$.

Let $C_i^*\left(\vec{A}\right) = C^*\left(\vec{A}\right) \cup \left\{x \in \mathcal{D}(\vec{A}_{\omega \cdot i}) | \forall n \neg C\,(x,n)\right\}$. Then whenever $B(x,i)$, $x \in A \Leftrightarrow x \in C_i^*(\vec{A})$. Analogous to the proof of Theorem 2.9, obtain the following:

(i) T is the tree with ordinal auxiliary moves λ_i for $\vec{D}$.

(ii) For $y \in [T]$, obtain $f(y)$ from y by deleting the λ_i's.

(iii) I wins $G\left(A_T, T\right) \Leftrightarrow$ either II badly loses or noone badly loses and $f(y) \in A$.

(iv) $T' = \{p \in T | \exists y \in [T_p]$ such that y is not badly lost$\}$.

(v) $\left(\tilde{T}, \pi, \phi\right) \in L\left[\vec{\#}^{k\infty}\right]$ is a $L[\vec{\#}^{k\infty}]$-covering of T that unravels each

$$B_i = \{y \in [T'] \,|B\left(f(y), i\right)\}.$$

For $z \in \tilde{T}$:

$z \in \pi^{-1}(A_T) \Leftrightarrow$ either $\pi(z)$ is badly lost for II or

$\pi(z)$ is not badly lost and $\exists i[z \in \pi^{-1}(B_i) \ \wedge \ f(\pi(z)) \in (\vec{C}_i)^*(\vec{A})]$.

Analogous to Lemma 2.8, we use the following:

Lemma 3.5.1. $G\left(\pi^{-1}\left(A_T\right), \tilde{T}\right)$ has a w.s. $s \in L\left[\vec{\#}^{k\infty}\right]$.

The proof of Lemma 3.5.1 is the same as that of Lemma 2.8 except we shall use w.s. for $C_i^*(\vec{A})$ in place of winning strategies for $\mathcal{D}(\vec{A}^p_{\omega\cdot i})$.

Let $\tilde{T}^p = \left\{q | p^\wedge q \in \tilde{T}\right\}$ and

$$E_i^p = \left\{y \in \left[\tilde{T}^p\right] | \pi\left(p^\wedge y\right) \in [T'] \text{ and } f\left(\pi\left(p^\wedge y\right)\right) \in C_i^*\left(\vec{E}\right)\right\}.$$

Let $p \in \tilde{T}$ have ordinal 0 iff either II badly loses on the λ_i's of $\pi(p)$ or noone badly loses on the λ_i's and for some $i \in \omega$, $\left[\tilde{T}_p\right] \subseteq B_i$ and player I has a w.s. for $G\left(E_i^p, \tilde{T}^p\right)$. Otherwise define the ordinal of a position as usual. By induction, if $\left[\tilde{T}_p\right] \subseteq B_i$, then $G\left(E_i^p, \tilde{T}^p\right)$ has a w.s. in $L\left(\#^{k\infty}(\tilde{T}_p)\right) \subseteq L[\vec{\#}^{k\infty}]$ since $\#^{k\infty}\left(\tilde{T}_p\right)$ exists and $C_i^*\left(\vec{E}\right) \in \left((k-1) * \Pi^1_1\right)^*_+$. Now Lemma 3.5.1 follows by the standard argument using the ordinal rank of positions. ■(Lemma 3.5.1)

By Lemma 3.5.1, $\phi(s) \in L\left[\vec{\#}^{k\infty}\right]$ is a w.s. for $G\left(A_T, T\right)$. Integrate $\phi(s)$ respect to the λ_i's and obtain a w.s. in $L(\#^{(k+1)\infty}(0))$ for the game A. ■(Theorem 3.5)

We now wish to show Theorem 3.5 for $k < \omega_1^{CK}$.

Theorem 3.6. If $k < \omega_1^{CK}$ and $L\left[\vec{\#}^{k\infty}\right]$ has indiscernibles, then $\mathrm{Det}\left(k * \Pi^1_1\right)^*_+$.

Proof: The proof is by induction on k and the proof for k a successor ordinal is identical to the proof of Theorem 3.5. So we shall assume k is a limit ordinal.

Let $\vec{B} = \langle B_\alpha | \alpha < k\rangle$, $\vec{A}$, and $\vec{D} = \langle D_\alpha | \alpha \leq \omega \cdot m\rangle$ witness that $A \in \left(\alpha * \Pi^1_1\right)^*_+$, where

$$x \in \bigcap_{\alpha \leq \omega\cdot m} D_\alpha \Leftrightarrow \exists n\left((\vec{B})(x, n)\right).$$

Such a $\vec{D}$ exists by the proof of Theorem 3.3 since $\exists n\left((\vec{B})(x,n)\right) \Leftrightarrow \exists n(\bigcup_{\alpha<k} B_\alpha(x,n))$. Let $\tilde{B}(x,n) \Leftrightarrow B_{(n)_0}(x,(n)_1)$, where $n \mapsto ((n)_0,(n)_1)$ is a recursive bijection from ω onto $k \times \omega$. Let B uniformize $\tilde{B}$.

Follow the proof of Theorem 3.5, so that we have (i) through (v) as stated in that proof. For $z \in \tilde{T}$:

$z \in \pi^{-1}(A_T) \Leftrightarrow$ either $\pi(z)$ is badly lost for II or

$\pi(z)$ is not badly lost and $\exists i[z \in \pi^{-1}(B_i) \ \wedge \ f(\pi(z)) \in (\vec{B}_{(i)_0})^*\left(\vec{A}, \mathcal{D}(\vec{A}_{\omega\cdot(i)_1})\right)$.

$\pi^{-1}(B_i)$ is clopen, and by Induction Hypotheses, $(\vec{B}_{(i)_0})^* \left(\vec{A}, \mathcal{D}(\vec{A}_{\omega\cdot(i)_1})\right)$ has a w.s. in $L(\#^{((i)_0+1)\infty}(0))$. Therefore, by the proof of Lemma 3.5.1, $G(\pi^{-1}(A_T), \tilde{T})$ has a w.s. $s \in L[\vec{\#}^{k\infty}]$. Integrate the w.s. $\phi(s) \in L[\vec{\#}^{k\infty}]$ for $G(A_T, T)$ with respect to the ξ_i's and obtain a w.s. in $L(\#^{(k+1)\infty}(0))$ for the game A. ■(Theorem 3.6)

Remark. $k < \omega_1^{CK}$ was important in that this was used to have $C \subseteq B$ in Lemma 3.3; specifically we used that the union of less than ω_1^{CK} Π_1^1 sets is Π_1^1. $C \subseteq B$ in turn was used to get that a play not badly lost is in $\pi^{-1}(A_T)$ only if it is in one of the clopen sets $\pi^{-1}(B_i)$. Clearly we can generalize Theorem 3.6 for larger k if we only consider $(k * \Pi_1^1)_+^*$ sets $(\langle B_\alpha | \alpha < k\rangle)^*(\vec{A}, \vec{D})$ in which for $i < j < k$, $B_i \subseteq B_j$.

By the natural generalization of the proof of the converse of Theorem 2.9, one can show the converse of Theorem 3.6 so that we have:

Theorem 3.7. Let $k < \omega_1^{CK}$. Then $L\left[\vec{\#}^{k\infty}\right]$ has indiscernibles iff $\mathrm{Det}\big(k * \Pi_1^1\big)_+^*$. ■

It should be clear from the proofs given in this paper that instead of relying on proving Theorem 3.6 by induction, we could for each $(k * \Pi_1^1)_+^*$ game $(\vec{B})^*(\vec{A}, \vec{D})$, instead define a corresponding open auxiliary game and construct (using the appropriate hypotheses) the desired w.s. for $(\vec{B})^*(\vec{A}, \vec{D})$ from a w.s. for that open game. This complicates the presentation. In the proof of Theorem 3.5, the induction hypotheses hid any uniformization of the C_j's in $C = (\langle C_j | j < k-1\rangle)$ by providing winning strategies for $C_i^*(\vec{A})$. Moreover, it is plausible that such uniformization of such C_j's need not be uniform, i.e. might depend on the particular $C_i^*(\vec{A})$. We strengthen Lemma 3.2 by uniformizing each "component" of the $(k * \Pi_1^1)$ set.

Theorem 3.8. (Normal Form Theorem for $(k * \Pi_1^1)^*$.) Let $k < \omega_1^{CK}$. If $A \in (k * \Pi_1^1)^*$, then there exist $\vec{B} = \langle B_i | i < k\rangle$ and $\vec{E}$ which witness that $A \in (k * \Pi_1^1)^*$ and such that each B_i is uniformized and $\bigcap_{\alpha < \omega^2} E_\alpha = \emptyset$.

Proof for successor case: Let $C \in (k * \Pi_1^1)$, C_k, and $\vec{A}$ witness that $A \in ((k+1) * \Pi_1^1)^*$. By induction, assume there exist $\vec{C} = \langle C_\beta | \beta < k\rangle$ and $\vec{F}$ such that $(\vec{C})^*(\vec{F}) = C^*(\vec{A})$, each

C_i is uniformized (for $i < k$), and $\bigcap_{\alpha<\omega^2} F_\alpha = \emptyset$. Let B_k uniformize $\bigcup_{i\leq k} C_i$. Let

$$x \in E_{\omega\cdot n+1} \Leftrightarrow \text{either } [\exists m > n B_k(x,m) \text{ and } (x \in A_{\omega\cdot n+i} \text{ or } \exists j \leq n (x,j) \in \bigcup_{i\leq k} C_i] \\ \text{or } [\exists m \leq n (B_k(x,m) \text{ and } x \in F_{\omega\cdot(n-m)+i})].$$

For fixed x, since at most one m exists such that $B_k(x,m)$ and $\bigcap_{\alpha<\omega^2} F_\alpha = \emptyset$, $x \notin \bigcap_{\alpha<\omega^2} E_\alpha$. Hence $\bigcap_{\alpha<\omega^2} E_\alpha = \emptyset$.

For $i < k$, let $B_i(x,n) \Leftrightarrow \exists m \leq n (B_k(x,m) \wedge C_i(x, n-m))$. Let $\vec{B} =_{df} \langle B_i | i \leq k\rangle$ and $\vec{E} =_{df} \langle E_\alpha | \alpha < \omega^2\rangle$. Then $(\vec{B})^*(\vec{E}) = (C, C_k)^*(\vec{A})$. ■(Successor Case)

Proof for k a limit: Let $\pi : \omega \to k$ be a recursive bijection. Let $\vec{C} = \langle C_i | i < k\rangle$ and $\vec{A}$ witness that A is $(k * \Pi^1_1)^*$. Let C uniformize $\{(x,i) | \exists n C_{\pi(i)}(x,n)\}$. For $i < k$, there exist $\vec{B}^i = \langle B^i_j | j < i\rangle$ and $\vec{F}^i = \langle F^i_\alpha | \alpha < \omega^2\rangle$ witnessing $(\vec{C}_{\pi(i)})^*(\vec{A})$ is $(\pi(i) * \Pi^1_1)^*$ and such that each B_i is uniformized and $\bigcap_{\alpha<\omega^2} F^i_\alpha = \emptyset$. For $\alpha < \omega^2$, let

$$x \in E_\alpha \Leftrightarrow \exists i (C(x,i) \text{ and } x \in F^i_\alpha).$$

For fixed x, since at most one i exists such that $C(x,i)$ and $\bigcap_{\alpha<\omega^2} F^i_\alpha = \emptyset$, $x \notin \bigcap_{\alpha<\omega^2} E_\alpha$. Hence $\bigcap_{\alpha<\omega^2} E_\alpha = \emptyset$.

Let $B_j(x,n) \Leftrightarrow \exists i [C(x,i) \wedge j < \pi(i) \wedge C_j(x,n)]$. Then $(\langle B_j | j < k\rangle)^*(\langle E_\alpha | \alpha < \omega^2\rangle) = A$. ■(Limit Case)

Recall B^i_j and F^i_α defined in the proof of Theorem 3.8 for k a limit ordinal. Let $B_k(x,i,j,n) \Leftrightarrow B^i_j(x,n)$, and let $F_k(x,i,j) \Leftrightarrow x \in F^i_{g(j)}$, where $g : \omega \to \omega^2$ is a fixed recursive bijection.

We left out details to the proof of Theorem 3.8 in that we need to select B_k and F_k which are Π^1_1. Further (to obtain this) we should instead be proving by induction that for each $k < \omega_1^{CK}$ and $A \in (k * \Pi^1_1)^*$ there exist Π^1_1 sequences $\langle B^i_j | i \leq k, j < i\rangle$ and $\langle F^i_\alpha | i \leq k, \alpha < \omega^2\rangle$ such that, setting $\vec{B}^i = \langle B^i_j | j < i\rangle$ and $\vec{F}^i = \langle F^i_\alpha | \alpha < \omega^2\rangle$, we have:

- $(\vec{B}^k)^*(\vec{F}^k) = A$.
- $(\vec{B}^i)^*(\vec{F}^i) = (\langle B^k_j | j < i\rangle)^*(\langle F^k_\alpha | \alpha < \omega^2\rangle)$.
- Each B^i_j is uniformized.

However, these changes would obscure the construction. Finally the successor case can be proved as the limit case was but this is somewhat artificial and the successor case is simplier.

By the proof of Theorem 3.3, we obtain a normal form for $(k{*}\Pi^1_1)^*_+$ sets from Theorem 3.8.

Theorem 3.9. (Normal Form Theorem for $(\Pi^1_1)^*_+$.) Let $k < \omega_1^{CK}$. If $A \in (k{*}\Pi^1_1)^*_+$, then there exist $\vec{B} = \langle B_i | i < k\rangle$, $\vec{E}$, and $\vec{D} = \langle D_\alpha | \alpha < \omega \cdot m\rangle$ which witness that $A \in (k * \Pi^1_1)^*_+$ and such that each B_i is uniformized, $\bigcap_{\alpha<\omega^2} E_\alpha = \emptyset$, and $x \in \bigcap_{\alpha \le \omega\cdot m} D_\alpha \Leftrightarrow \exists n \exists i < k B_i(x, n)$.

By induction, it is easy to show the classes of this section are $\Delta(\omega^2 - \Pi^1_1)$.

Theorem 3.10. For $k < \omega_1^{CK}$, $(k * \Pi^1_1)^* \subseteq (k * \Pi^1_1)^*_+ \subseteq ((k+1) * \Pi^1_1)^* \subseteq \Delta(\omega^2 - \Pi^1_1)$.

Proof: Since $(k * \Pi^1_1)^* \subseteq (k * \Pi^1_1)^*_+$ it is enough to prove that $(k * \Pi^1_1)^*_+ \subseteq \Delta(\omega^2 - \Pi^1_1)$ by induction on k. Let $\vec{B} = \langle B_\alpha | \alpha < k\rangle$, $\vec{A}$, and $\vec{D} = \langle D_\alpha | \alpha < \omega \cdot m\rangle$ witness that $(\vec{B})^*(\vec{A}, \vec{D}) \in (k * \Pi^1_1)^*_+$. By induction, we may assume that for each $\alpha < k$ and $m < \omega$, there exists $\langle F^{\alpha,m}_\beta | \beta < \omega^2\rangle$ which witnesses that the $(\alpha * \Pi^1_1)^*_+$ set $(\vec{B}_\alpha)^*(\vec{A}, \mathcal{D}(\vec{A}_{\omega\cdot m}))$ is $\Delta(\omega^2 - \Pi^1_1)$.

Successor Case: Let $\tilde{B}$ uniformize $\bigcup_{\alpha<k} B_\alpha \in \Pi^1_1$. Let

$$x \in E_{\omega\cdot n+i} \Leftrightarrow \text{either } \exists \ell > n[\tilde{B}_{k-1}(x,\ell) \text{ and } (x \in A_{\omega\cdot n+i} \text{ or } \exists j \le n (x,j) \in \bigcup_{\alpha<k} B_\alpha)]$$
$$\text{or } \exists \ell \le n[\tilde{B}_{k-1}(x,\ell) \text{ and } x \in F^{k-1,m}_{\omega\cdot(n-\ell)+i}].$$

Limit Case: Let $\pi : \omega \to k$ be a recursive bijection. Let $\tilde{B}$ uniformize $\{(x,i) | B_{\pi((i)_0)}(x,(i)_1)\}$. Let

$$x \in E_\beta \Leftrightarrow \exists \ell[\tilde{B}(x,\ell) \text{ and } x \in F_\beta^{\pi((\ell)_0),(\ell)_1}].$$

In both cases, $\bigcap_{\beta<\omega^2} E_\beta = \emptyset$ since for each $\alpha < k$ and $m < \omega$, $\bigcap_{\beta<\omega^2} F^{\alpha,m}_\beta = \emptyset$. Also $(\vec{B})^*(\vec{A}) = \mathcal{D}(\langle E_\beta | \beta < \omega^2\rangle)$.

Let $E'_\beta = D_\beta \cup E_0$ for $\beta < \omega \cdot m$ and $E'_{\omega\cdot(n+m)+i} = E_{\omega\cdot n+i}$. Then $\langle E'_\alpha | \alpha < \omega^2\rangle$ witnesses that $(\vec{B})^*(\vec{A}, \vec{D})$ is $\Delta(\omega^2 - \Pi^1_1)$. ▪(Theorem 3.10)