Section 1. The relevent classes.

In this section, we review the definition of the relevant classes $(\Pi^1_1)^*$ and $(\Pi^1_1)^*_+$, and prove the needed normal form for such sets so as to carry out the proof of the Main Theorem. The basic definitions are discussed in more detail in §1 of [Du4] but the presentation here is complete.

Definition 1.1. The difference kernel. Let $\vec{A}$ be a sequence $\langle A_\alpha|\alpha<\beta\rangle$ of subsets of ω^ω. We shall denote the the *difference kernel* of $\vec{A}$ by $\mathcal{D}(\vec{A})$, so that:

$$x\in\mathcal{D}(\vec{A})\Leftrightarrow\mu\alpha(x\notin A_\alpha \text{ or } \alpha=\beta) \text{ is odd.}$$

The notion of difference kernel is discussed in [Hd]. Now we review the definition of $\beta-\Pi^1_1$ (see Definition 1.3 of [Du4] for more detail).

Definition 1.2. $\beta-\Pi^1_1$. For $\beta\in ON$, $\beta-\mathbf{\Pi^1_1}$ is the collection of difference kernels $\mathcal{D}(\langle E_\alpha|\alpha<\beta\rangle)$ in which each E_α is $\mathbf{\Pi^1_1}$. For β a recursive ordinal, A is (lightface) $\beta-\Pi^1_1$ iff A is the difference kernel of a Π^1_1 sequence $\vec{A}=\langle A_\alpha|\alpha<\beta\rangle$. In this case, we say that $\vec{A}$ [respectively, $\langle A_\alpha|\alpha\le\beta\rangle$, where $A_\beta=\emptyset$] *witnesses* that A is $\beta-\Pi^1_1$. A is $\Delta(\Gamma)$ iff both A and its complement are Γ.

If $\vec{A}=\langle A_\alpha|\alpha<\beta\rangle$ and $\gamma\le\beta$, we let $\vec{A}_\gamma =_{df} \langle A_\alpha|\alpha<\gamma\rangle$ (whereas $\vec{\#}^{k\infty} =_{df} \langle\#^{(\beta+1)\infty}|\beta<k\rangle$).

Proposition 1.3. [Du4] A is $\Delta(\omega^2-\Pi^1_1)$ iff there exists a sequence $\vec{A}=\langle A_\alpha|\alpha<\omega^2\rangle$ which witnesses that A is $\omega^2-\Pi^1_1$ and with empty intersection, i.e. $\bigcap\limits_{\alpha<\omega^2} A_\alpha=\emptyset$. In this case, we say $\vec{A}$ *witnesses* that A is $\Delta(\omega^2-\Pi^1_1)$. ■

We are interested in classes of sets near the "bottom" of $\Delta(\omega^2-\Pi^1_1)$. For the remainder of this paper, we reserve the notation $\vec{A}$ to denote an ω^2 sequence of sets of reals. Define $n(x)$ to be the least $n\in\mathbf{N}$ such that $\mu\alpha(x\notin A_\alpha)$ exists and is $<\omega\cdot n$. If $\vec{A}$ is as in Proposition 1.3, the function $n:\omega^\omega\to\mathbf{N}$ is total. We shall consider classes of sets for which n is "simple."

Definition 1.4. $(\Gamma)^*$. If n is a total function from the reals into $\mathbf{N}$, let

$$n^*(\vec{A}) =_{df} \{x|x\in\mathcal{D}(\vec{A}_{\omega\cdot n(x)})\}.$$

For $B \subseteq (\omega^\omega) \times \omega$, let:

$$n_B(x) =_{df} \begin{cases} \mu n(B(x,n)) & \text{if } \exists n B(x,n) \\ 0 & \text{otherwise.} \end{cases}$$

$$B^*(\vec{A}) =_{df} n_B^*(\vec{A}).$$

For $\Gamma \subseteq \wp((\omega^\omega) \times \omega)$,

$$(\Gamma)^* =_{df} \{B^*(\vec{A}) | B \in \Gamma \text{ and } \vec{A} \text{ witnesses that some set is } \omega^2 - \Pi^1_1\}.$$

$$x \in B^*(\vec{A}) \Leftrightarrow x \in \mathcal{D}(\vec{A}_{\omega \cdot n_B(x)})$$

$$\Leftrightarrow \exists n[B(x,n) \ \wedge \ \forall m < n \neg B(x,m) \ \wedge \ \exists \alpha < \omega \cdot n(\alpha \text{ is odd} \wedge x \in \bigcap_{\beta<\alpha} A_\beta \setminus A_\alpha)].$$

For $x \in B^*(\vec{A})$, not only must $\mu\alpha(x \notin A_\alpha)$ be odd (and exist), α must be less than $\omega \cdot n_B(x)$ (that is, x must also "fall out" of $\vec{A}$ by stage $n_B(x)$). When $n_B(x)$ is constant and equals m, $B^*(\vec{A})$ is $\omega \cdot m - \Pi^1_1$. We are interested in a class slightly larger than $(\Pi^1_1)^*$.

Definition 1.5. $(\Gamma)^*_+$. If $B \subseteq ({}^\omega\omega) \times \omega$, $\vec{A}$ witnesses some set is $\omega^2 - \Pi^1_1$, and $D \in \bigcup_{\beta<\omega^2} \beta - \Pi^1_1$, let $B^*\left(\vec{A}, D\right) =_{df} B^*\left(\vec{A}\right) \cup \{x \in D | \forall n \neg B(x,n)\}$; if $\vec{D}$ witnesses that $D \in \bigcup_{\beta<\omega^2} \beta - \Pi^1_1$ (i.e. $D = \mathcal{D}(\vec{D})$), then $B^*\left(\vec{A}, \vec{D}\right) =_{df} B^*\left(\vec{A}, D\right)$.

For $\Gamma \subseteq \wp\left(({}^\omega\omega) \times \omega\right)$, let

$$E \in (\Gamma)^*_+ \ \Leftrightarrow E = B^*\left(\vec{A}, D\right) \text{ for some } B \in \Gamma,$$
$$\vec{A} \text{ witnesses some set is } \omega^2 - \Pi^1_1, \text{and } D \in \bigcup_{\beta<\omega^2} \beta - \Pi^1_1.$$

In this case, we say that B, $\vec{A}$ and D [respectively $\vec{D}$] *witness* that $B^*\left(\vec{A}, D\right)$ [respectively $B^*\left(\vec{A}, \vec{D}\right)$] is $(\Gamma)^*_+$. △

$(\Pi^1_1)^*_+$ lies "near the bottom" of $\Delta(\omega^2 - \Pi^1_1)$, even though this is not at first obvious.

Proposition 1.6. (Proposition 1.12 of [Du4]) $\left(\Pi^1_1\right)^* \subseteq \left(\Pi^1_1\right)^*_+ \subseteq \Delta\left(\omega^2 - \Pi^1_1\right)$.

Both inclusions of Proposition 1.6 are in fact proper (see Theorem 1.17 and 3.10). The proof of Proposition 1.6 gives us the normal form for $(\Pi^1_1)^*_+$ sets which will be needed to invoke quasi-Borel determinacy in the proof of the Main Theorem 2.9.

Lemma 1.7. (Normal Form Lemma for $(\Pi^1_1)^*_+$) Let A be $(\Pi^1_1)^*_+$. Then (for some $m < \omega$) there exists $B \in \Pi^1_1$, $\vec{E} = \langle E_\alpha | \alpha < \omega^2 \rangle$, and $\vec{D} = \langle D_\alpha | \alpha \leq \omega \cdot m \rangle$ which witness that A is $(\Pi^1_1)^*_+$ and such that:

(i) (Uniformization) B is *uniformized*, i.e. for each x there exists at most one m such that $B(x, m)$.

(ii) $\exists n B(x, n) \Leftrightarrow \forall \alpha \leq \omega \cdot m (x \in D_\alpha)$.

In this case, we shall say that B, $\vec{E}$, and $\vec{D}$ *strongly witness* that $A \in (\Pi^1_1)^*_+$. We now simultaneously prove Proposition 1.6 and the Normal Form Lemma 1.7.

Proof: Let $\hat{B} \in \Pi^1_1$, $\vec{A} = \langle A_\alpha | \alpha < \omega^2 \rangle$, and $D \in \omega \cdot m - \Pi^1_1$ witness that $B^* \left(\vec{A}, D \right)$ is $\left(\Pi^1_1 \right)^*_+$. By Theorem 4E.4 on page 235 of [Mo], let B uniformize $\hat{B}$ so that (i) holds and

- $\exists n B(x, n) \Leftrightarrow \exists n \hat{B}(x, n)$.

Let

$$x \in E_{\omega \cdot k + i} \Leftrightarrow \left(\exists n > k \hat{B}(x, n) \text{ and } [x \in A_{\omega \cdot k + i} \text{ or } \exists \ell \leq k B(x, \ell)] \right).$$

Let $\langle d_\alpha | \alpha < \omega \cdot m \rangle$ witness that $D \in \omega \cdot m - \Pi^1_1$. For $\alpha < \omega \cdot m$, let

$$F_\alpha =_{df} D_\alpha =_{df} \{ x | x \in d_\alpha \text{ or } \exists n B(x, n) \};$$

let $x \in D_{\omega \cdot m}$ iff $\exists n B(x, n)$; and let $F_{\omega \cdot m + \alpha} = E_\alpha$. Then $\langle F_\alpha | \alpha < \omega^2 \rangle$ witnesses that $B^*(\vec{A}, D)$ is $\Delta \left(\omega^2 - \Pi^1_1 \right)$, proving Proposition 1.6.

Also, B, $\vec{E}$, and $\vec{D} = \langle D_\alpha | \alpha \leq \omega \cdot m \rangle$ satisfy the Normal Form Lemma, i.e. strongly witnesses that A is $(\Pi^1_1)^*_+$. ■(Proposition 1.6 and Lemma 1.7)

Remark 1. $\vec{E} = \langle E_\alpha | \alpha < \omega^2 \rangle$ witnesses that $\hat{B}^*(\vec{A})$ is $\Delta \left(\omega^2 - \Pi^1_1 \right)$.

Remark 2. Let $\tilde{B}(x, n) \Leftrightarrow n \geq m \wedge B(x, n - m)$. Then $\tilde{B}$, $\vec{F}$, and $\vec{D}$ also strongly witness that $A \in (\Pi^1_1)^*_+$. Since $A = (\tilde{B}, \{(x, n) | n = m\})^*(\vec{F})$, $(\Pi^1_1)^*_+ \subseteq (\Pi^1_1, \Sigma^0_1)^* \subseteq (\Pi^1_1, \Pi^1_1)^*$. One can show $(\alpha * \Pi^1_1)^*_+ \subseteq (\beta * \Pi^1_1)^* \subseteq \Delta \left(\omega^2 - \Pi^1_1 \right)$ for $\alpha < \beta < \omega_1^{CK}$.

We have already provided the needed material from this section for proving the Main Theorem 2.9 (so some readers may wish to skip to the next section).

$(\Pi^1_1)^*$ is strictly smaller than $(\Pi^1_1)^*_+$ (see Theorem 1.17), even though it is conjectured that they have the same determinacy strength. To show the former, we first investigate the dual class of $(\Pi^1_1)^*$.

Definition 1.8. $(\Gamma)^*_\lambda$. If B and $\vec{A}$ witness some set E is $(\Pi^1_1)^*$, let $B^*_{II}(\vec{A}) =_{df} B^*(\vec{A})$ and

$$B^*_I(\vec{A}) =_{df} B^*(\vec{A}) \cup \{x|\forall n \neg B(x,n)\}.$$

Let $(\Gamma)^*_{II} =_{df} (\Gamma)^*$ and $(\Gamma)^*_I$ be the collection of sets $B^*_I(\vec{A})$ where $B \in \Gamma$ and $\vec{A}$ witness $B^*(\vec{A})$ is $(\Gamma)^*$. For $\lambda = I, II$, we define $B \in \Gamma$ and $\vec{A}$ *witness* [respectively, *strongly witness* that $B^*_\lambda(\vec{A})$ is $(\Gamma)^*_\lambda$ to have the obvious meaning; we require the B to be uniformized in the definition of strongly witness. △

If $\forall n \neg B(x,n)$, then x is a win for player λ in the game $B^*_\lambda(\vec{A})$. If there exists a least n such that $B(x,n)$, then

$$x \in B^*_I(\vec{A}) \Leftrightarrow x \in B^*_{II}(\vec{A}) \Leftrightarrow x \text{ falls out of } \vec{A} \text{ for player I by stage } n, \text{ i.e. } x \in \mathcal{D}(\vec{A}_{\omega\cdot n}).$$

By the proof of the Normal Form Lemma 1.7, we have:

Lemma 1.9. (Normal Form Lemma for $(\Pi^1_1)^*_\lambda$.) For $\lambda \in I, II$, every $(\Pi^1_1)^*_\lambda$ set is witnessed to be such by some uniformized $B \in \Pi^1_1$ and $\vec{A}$. ■

Theorem 1.10. $(\Pi^1_1)^*_I$ and $(\Pi^1_1)^*_{II}$ are dual classes of one another.

Proof: Let B and $\vec{A} = \langle A_\alpha | \alpha < \omega^2 \rangle$ witness $B^*_I(\vec{A})$ is $(\Pi^1_1)^*_I$. By the Normal Form Lemma, wlog we may assume $\forall x \exists$ at most one n such that $B(x,n)$. Let:

$$C(x,n) \Leftrightarrow n \geq 1 \text{ and } B(x,n-1).$$

$$x \in E_{\omega\cdot n+k+3} \Leftrightarrow x \in A_{\omega\cdot n+k}, \quad x \in E_{\omega\cdot n} \Leftrightarrow \exists \ell \geq n B(x,\ell),$$

$$x \in E_{\omega\cdot n+1} \Leftrightarrow \exists \ell > n B(x,\ell), \quad x \in E_{\omega\cdot n+2} \Leftrightarrow x = x$$

So $C(x,n) \Rightarrow B(x,n-1) \Rightarrow \forall \ell \geq n \neg B(x,\ell) \Rightarrow x \notin E_{\omega\cdot(n-1)+1} \Rightarrow \exists \alpha < \omega \cdot n (x \notin E_\alpha)$.

$$\begin{aligned} x \in B^*_I(\vec{A}) &\Leftrightarrow \exists n [B(x,n) \;\&\; \mu\alpha(x \notin A_\alpha) \text{ is odd and } < \omega \cdot n] \text{ or } \forall n \neg B(x,n) \\ &\Leftrightarrow \exists n [B(x,n) \;\&\; \mu\alpha(x \notin E_\alpha) \text{ is even and } < \omega \cdot (n+1)] \text{ or } \forall n \neg B(x,n) \\ &\Leftrightarrow x \notin C^*_{II}(\vec{E}), \text{ since } C(x,n) \Rightarrow \exists \alpha < \omega \cdot n (x \notin E_\alpha). \end{aligned}$$

Thus, $(\Pi^1_1)^*_I \subseteq$ dual class of $(\Pi^1_1)^*_{II}$. Similarly show $(\Pi^1_1)^*_I \subseteq$ dual class of $(\Pi^1_1)^*_I$. ■(Theorem 1.10)

Corollary 1.11. $\mathrm{Det}(\Pi^1_1)^*_I \Leftrightarrow \mathrm{Det}(\Pi^1_1)^*_{II}$. ■

Recall that it is conjectured that $\mathrm{Det}(\Pi^1_1)^*$ implies $\mathrm{Det}(\Pi^1_1)^*_+$. Of course one might try to show the conjecture false by producing a $(\Pi^1_1)^*_+$ set $A = B^*(\vec{A}, D)$ whose determinacy may fail, assuming $\mathrm{Det}(\Pi^1_1)^*$. By Corollary 1.11, A cannot be $(\Pi^1_1)^*_\lambda$ for both $\lambda = I, II$. Therefore, $D \notin \{\emptyset, \omega^\omega\}$, since otherwise $B^*(\vec{A}, D) = B^*_\lambda(\vec{A})$ for some $\lambda \in \{I, II\}$.

Theorem 1.12. $(\Pi^1_1)^*_I \cap (\Pi^1_1)^*_{II} = (\Delta^1_1)^*$.

Proof Clearly $(\Delta^1_1)^* \subseteq (\Pi^1_1)^*_I \cap (\Pi^1_1)^*_{II}$. So let $A \in (\Pi^1_1)^*_I \cap (\Pi^1_1)^*_{II}$, let B and $\vec{A}$ witness that $A = B^*(\vec{A}) \cup \{x | \forall n \neg B(x,n)\} \in (\Pi^1_1)^*_I$, and let C and $\vec{E}$ witness that $A = C^*(\vec{E}) \in (\Pi^1_1)^*$. Let $D(x,n) \Leftrightarrow_{df} B(x,n) \vee C(x,n)$. Let $x \in F_\alpha \Leftrightarrow x \in E_\alpha$ or ($\exists n B(x,n)$ and $x \in A_\alpha$). Then $D^*(\vec{F}) = C^*(\vec{E}) = A$.

<u>Claim</u> $\forall x \exists n D(x,n)$.

Pick x. If $\exists n B(x,n)$, then $\exists n D(x,n)$. So assume $\forall n \neg B(x,n)$. Then $x \in B^*(\vec{A}) \cup \{x | \forall n \neg B(x,n)\} = A = D^*(\vec{F})$. Hence $\exists n D(x,n)$. ■(Claim)

By the Normal Form Lemma, $\exists \tilde{D}$ and $\vec{G}$ which witness $\tilde{D}^*(\vec{G}) \in (\Pi^1_1)^*$ such that $\tilde{D}^*(\vec{G}) = D^*(\vec{F})$ and (\$) $\tilde{D}$ uniformizes D. Since $\forall x \exists n D(x,n)$, by (\$) $\forall x \exists ! n \tilde{D}(x,n)$. Hence $\neg \tilde{D}(x,n) \Leftrightarrow \exists m \notin n \tilde{D}(x,m)$, so $\tilde{D} \in \Delta^1_1$ and $A = \tilde{D}^*(\vec{G}) \in (\Delta^1_1)^*$. ■(Theorem 1.12)

Theorem 1.12 generalizes to give:

Theorem 1.13. If $\Gamma \subseteq \Pi^1_1$, $(B, C \in \Gamma \Rightarrow B \vee C \in \Gamma)$, and $\mathrm{Unif}(\Gamma)$, then $\Delta((\Gamma)^*) = (\Gamma)^*_I \cap (\Gamma)^*_{II} = (\Delta(\Gamma))^*$. ■

Corollary 1.14. If $(\Pi^1_1)^*_I \subseteq (\Pi^1_1)^*$ or $(\Pi^1_1)^* \subseteq (\Pi^1_1)^*_I$, then $(\Pi^1_1)^* = (\Pi^1_1)^*_I = (\Delta^1_1)^*$. Hence, if $(\Pi^1_1)^* = (\Pi^1_1)^*_+$, then $(\Pi^1_1)^*_+ = (\Pi^1_1)^* = (\Pi^1_1)^*_I = (\Delta^1_1)^*$.

Proof: The second statement follows from the first and from $(\Pi^1_1)^*_I \subseteq (\Pi^1_1)^*_+$.

Assume one of $(\Pi^1_1)^*_I$ and $(\Pi^1_1)^*_{II}$ contains the other. Then by Theorem 1.12, one of these classes equals $(\Delta^1_1)^*$, and by Theorem 1.10, the other equals the dual of $(\Delta^1_1)^*$. But $(\Delta^1_1)^* = (\Delta^1_1)^*_+$ is self-dual. Thus, $(\Pi^1_1)^* = (\Pi^1_1)^*_{II} = (\Delta^1_1)^*$. ∎(Corollary 1.14)

$SH(\#^\infty)$ (i.e. $L[\#^\infty] \models$ "$\#^\infty$ is total") is strictly weaker than $\mathrm{Det}(\Pi^1_1)^*$ by Theorem 1(iii) but stronger than $\mathrm{Det}(\Delta^1_1)^*$ by Theorem 1(ii) and/or 2(ii). Therefore, $\mathrm{Det}(\Delta^1_1)^*$ is strictly weaker than $\mathrm{Det}(\Pi^1_1)^*$ and $(\Delta^1_1)^*$ is properly contained in $(\Pi^1_1)^*$ so that by Propositions 1.10 and 1.12, we have:

Theorem 1.17. $(\Delta^1_1)^* = (\Delta^1_1)^*_+ = \Delta((\Pi^1_1)^*)$ is properly contained in each of $(\Pi^1_1)^*$ and its dual $(\Pi^1_1)^*_I$, both of which are properly contained in $(\Pi^1_1)^*_+$. ∎