Determinacy and Extended Sharp Functions on the Reals
Part I : Obtaining Determinacy from Extended Sharps

Derrick Albert DuBose

Abstract. We characterize in terms of determinacy, the existence of several inner models of "every real r has a sharp $r^{\#}$. For each $\beta, \gamma \in \mathbf{N}$, we inductively define the extended sharp $r^{\beta \#^1_\gamma}$ of reals r so that $r^{1\#^1_1} = r^{\#}$, and we characterize in terms of determinacy the existence of several inner models of "$r^{\beta\#^1_\gamma}$ exists for every real r."

Let $\#^1_1$ be the partial sharp function on the reals and inductively define $\#^1_{\gamma+1}$ to be the iterated partial sharp function on the reals that maps x to another real $x^{\#^1_{\gamma+1}}$ which codes indiscernibles for $L(x)[\#^1_\gamma]$. For $\gamma \in \mathbf{N}$, we define two classes of sets, $(\gamma * \Pi^0_1)^*$ and $(\gamma * \Pi^0_1)^*_+$, which lie strictly between $\bigcup_{\beta<\omega^2}(\beta - \Pi^1_1)$ and $\Delta(\omega^2 - \Pi^1_1)$. We show that the determinacy of $(\gamma * \Pi^0_1)^*$ follows from $L[\#^1_\gamma] \models$"$x^{\#^1_\gamma}$ exists for every real x." We also show that the existence of indiscernibles for $L[\#^1_\gamma]$ implies a slightly stronger determinacy hypothesis, the determinacy of $(\gamma * \Pi^0_1)^*_+$, and prove the converse in Part II of this paper.

We generalize the above results. If $L(x^{\beta\#^1_{\gamma+1}})[\#^1_\gamma]$ has indiscernibles, we let

$$x^{1\#^1_{\gamma+1}} = x^{\#^1_{\gamma+1}} \text{ and } x^{(\beta+1)\#^1_{\gamma+1}} = (x^{\beta\#^1_{\gamma+1}})^{\#^1_{\gamma+1}}.$$

For each $\beta \in \omega$, we define the two classes of sets $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ and $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$, which lie between $(\gamma * \Pi^0_1)^*_+$ and $((\gamma+1) * \Pi^0_1)^*$. We show that all $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ games are determined if both $0^{\beta\#^1_{\gamma+1}}$ exists and

$$L(0^{\beta\#^1_{\gamma+1}})[\#^1_\gamma] \models \text{"}x^{\#^1_\gamma} \text{ exists for every real } x.\text{"}$$

We also show that the existence of $0^{(\beta+1)\#^1_{\gamma+1}}$ implies the slightly stronger determinacy hypothesis, Det($(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$). We prove the converse in Part II of this paper.

Introduction. For any collection A of functions $f \in {}^\omega\omega$, we associate a two-person, infinite game which we denote by either G_A or $G(A)$:

I	f(0)	f(2)	f(4)	. . .
II	f(1)	f(3)	f(5)	.

The game G_A has two players, I and II, who alternately choose elements of ω. First player I chooses $f(0) \in \omega$ and then player II chooses $f(1) \in \omega$. In general, once $f(0), f(1), f(2), ..., f(2n-1)$ have been chosen, player I chooses $f(2n)$ and then player II chooses $f(2n+1)$. Each $f(2n)$ is called a *move* of player I; whereas, $f(2n+1)$ is a *move* of player II. Player I wins G_A if $f \in A$, whereas, player II wins G_A if $f \notin A$. *Strategy* and *winning strategy* (abbreviated w.s.) for player I [resp. II] have the natural meanings–we refer the reader to Chapter Six of [Mo] for the basic notions associated with games. We say that the game G is determined if one of the players has a w.s., and we denote this by Det(G). Similarly, if Det(G_A) for any $A \in \Gamma$, then we denote this by Det(Γ).

Recently, Martin [Ma3] has shown that the determinacy of $\Delta((\omega^2+1)-\Pi^1_1)$ follows from the existence of L_μ. Initially Solovay and then Friedman, Martin, and Solovay proved results that established the existence of L_μ follows from "low" levels of projective determinacy (e.g. Σ^1_3, Δ^1_2, $(\omega^2{+}1){-}\Pi^1_1$). By the middle 1970's, Martin showed that the determinacy of $\omega^2{-}\Pi^1_1$ follows from the existence of L_μ. By the late 1970's, the existence of $0^\#$ had been characterized in terms of determinacy. Friedman [Fr] showed that the existence of $0^\#$ implies the determinacy of $3{-}\Pi^1_1$ games. Then Martin showed that all $\bigcup_{\beta<\omega^2}(\beta{-}\Pi^1_1)$ games are determined iff $0^\#$ exists. In 1975, Martin showed that Det($3-\Pi^1_1$) implies $0^\#$ exists; soon after, Harrington [Ha] showed that the determinacy of Π^1_1 games implies the existence of $0^\#$. Thus, $0^\#$ exists iff Det($\beta-\Pi^1_1$) for some (all) $\beta < \omega^2$.

In [Du1], DuBose characterizes the existence of $0^{\#\#}$, $0^{\#\#\#}$, $0^{\#\#\#\#}$, ... in terms of determinacy. He defines for each $k \in \omega$, two classes of sets, $(k*\Sigma^0_1)^*$ and $(k * \Sigma^0_1)^*_+$, which lie strictly between $\bigcup_{\beta<\omega^2}(\beta - \Pi^1_1)$ and $\Delta(\omega^2 - \Pi^1_1)$. He shows that if we define $0^{k\#}$ to be the k^{th} iterated sharp (i.e. let $0^{1\#}$ be $0^{\#}$ and $0^{(k+1)\#}$ be $(0^{k\#})^{\#}$, then the existence of $0^{(k+1)\#}$ is equivalent to the determinacy of $((k+1) * \Sigma^0_1)^*$ as well as to the determinacy of $(k * \Sigma^0_1)^*_+$.

In [Du2], DuBose defines $\#_1$ as the (partial) sharp function on the reals (i.e. $\#_1(r) = r^{\#}$ whenever r is real such that $L(r)$ has indiscernibles). He then shows that the determinacy hypothesis, $\mathrm{Det}(\Pi^0_1)^*$, follows from

$$L[\#_1] \models \text{“every real has a sharp,”}$$

while a slightly stronger determinacy hypothesis, $\mathrm{Det}(\Pi^0_1)^*_+$, is equivalent to the existence of indiscernibles for $L[\#_1]$.

DuBose [Du3] generalizes the results found in [Du2]. He defines for each $k \in \omega$, $\#_k$ to be the (partial) sharp function on objects of type k and then relates the following to determinacy:

i.) $L[\#_k] \models$“every object of type k has a sharp,” and

ii.) indiscernibles for $L[\#_k]$ exist.

He shows that a certain determinacy hypothesis, $\mathrm{Det}(\Pi^0_k)^*$, follows from (i) and that the slightly stronger determinacy hypothesis, $\mathrm{Det}(\Pi^0_k)^*_+$, is equivalent to (ii). Each $(\Pi^0_k)^*$ and $(\Pi^0_k)^*_+$ are strictly smaller than $\Delta(\omega^2 - \Pi^1_1)$ but strictly larger than any $(i * \Sigma^0_1)^*$ or $(i * \Sigma^0_1)^*_+$.

In this paper, we also generalize the results of [Du2]. First we define several *extended* notions of the sharp functions on the reals, and for each notion of an extended sharp, we define several inner models of "every real has an extended sharp." Then we characterize each of these models in terms of determinacy.

Let $\#^1_1$ be the (partial) sharp function on the reals (i.e. $\#^1_1 = \#_1$). By induction on γ, define $\#^1_{\gamma+1}$ to be the iterated partial sharp function on the reals that maps x to another real $x^{\#^1_{\gamma+1}}$ which codes indiscernibles for $L(x)[\#^1_\gamma]$. For $\gamma \in \mathbf{N}$, we define two classes of sets, $(\gamma * \Pi^0_1)^*$ and $(\gamma * \Pi^0_1)^*_+$, which lie strictly between any $(\beta * \Sigma^0_1)^*_+$ and $(\Pi^0_2)^*$. We show that

iii.) the determinacy of $(\gamma * \Pi^0_1)^*$ follows from

$$L[\#^1_\gamma] \models \text{“}x^{\#^1_\gamma} \text{ exists for every real } x;\text{”}$$

and we show that

iv.) the existence of indiscernibles for $L[\#^1_\gamma]$ implies a slightly stronger determinacy hypothesis, the determinacy of $(\gamma * \Pi^0_1)^*_+$.

[Du4] proves the converse of (iv).

We also generalize (iii) and (iv). By induction on β, we define $x^{\beta\#^1_{\gamma+1}}$ as follows:

$$x^{1\#^1_{\gamma+1}} = x^{\#^1_{\gamma+1}} \text{ and } x^{(\beta+1)\#^1_{\gamma+1}} = (x^{\beta\#^1_{\gamma+1}})^{\#^1_{\gamma+1}}.$$

whenever $L(x^{\beta\#^1_{\gamma+1}})[\#^1_\gamma]$ has indiscernibles. For each $\beta \in \omega$, we define the two classes of sets

$$(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^* \text{ and } (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+,$$

which lie strictly between $(\gamma * \Pi^0_1)^*_+$ and $((\gamma+1) * \Pi^0_1)^*$. We show that

v.) if $0^{\beta\#^1_{\gamma+1}}$ exists and

$$L(0^{\beta\#^1_{\gamma+1}})[\#^1_\gamma] \models \text{“}x^{\#^1_\gamma} \text{ exists for every real } x\text{,”}$$

then all $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ games are determined. We also show that

vi.) the existence of $0^{(\beta+1)\#^1_{\gamma+1}}$ implies the slightly stronger determinacy hypothesis, $\mathrm{Det}((\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+)$. [Du4] proves the converse of (vi).

In Section One, we show (v) and (vi) for $\gamma = 1$. [Du2] shows (iii) and (iv) for $\gamma = 1$. In Section Two, (iii), (iv), (v), and (vi) are proved for any $\gamma \in \mathbf{N}$. In subsequent papers (e.g. [Du5]), we prove natural generalizations of (iii), (iv), (v), and (vi) with respect to objects of higher types than the reals.

Section Zero is the Preliminaries and consists of Sections 0.1, 0.2, 0.3, and 0.4. In Sections 0.1 and 0.2, we respectively define the models and the determinacy hypotheses used in this paper. In Section 0.3, we introduce some new terminology. In Section 0.4, we review the auxiliary games from [Du1,2].

The proofs in this paper use Martin's proof of Borel determinacy [Ma2] and his proof [Ma1] that Π^1_1 games are determined if L_μ exists. Furthermore, the determinacy and large cardinal hypotheses considered in this paper are each no weaker that $\mathrm{Det}(\Pi^0_1)^*$, but strictly weaker than $\mathrm{Det}(\Pi^0_2)^*$, which in turn is weaker than

$$L[\#_2] \models \text{“every set of reals has indiscernibles.”}$$

If $n \in \omega$, $k \in \omega$, and either $n < \gamma$ or $n = \gamma$ and $k < \beta$, then

$$(n * \Pi^0_1, k * \Sigma^0_1)^* \subset (n * \Pi^0_1, k * \Sigma^0_1)^*_+ \subset (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^* \subset (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+ \subset (\Pi^0_2)^*.$$

Similarly, the existence of $0^{k\#^1_n}$ is strictly weaker than the existence of $0^{\beta\#^1_\gamma}$ if either $n < \gamma$ or $n = \gamma$ and $k < \beta$. Therefore, the existence of any $0^{\beta\#^1_\gamma}$ such that $\beta \cdot \gamma > 1$ is stronger than the existence of $0^{1\#^1_1}$ (i.e. $0^\#$), but (as mentioned above) is weaker than

$$L[\#_2] \models \text{“every set of reals has indiscernibles,”}$$

which in turn is strictly weaker than the existence of L_μ. We illustrate the above relations in *Figures One* through *Four*.

Often we consider games in which the players' moves are not necessarily integers, but instead elements from some set X. Given

$$f(0), f(1), f(2), \ldots, f(n-1) \in X,$$

we may restrict the move $f(n)$ to be an element of some particular subset $X_{f(0),f(1),f(2),\cdots,f(n-1)}$ of X. In the case in which $X_{f(0),f(1),f(2),\cdots,f(n-1)} = \emptyset$, there is no possible $f(n)$ to be played and the player who was to play $f(n)$ loses. If $f(i) \in X_{f(0),f(1),f(2),\ldots,f(i-1)}$ for all $i < n$ and $X_{f(0),f(1),f(2),\ldots,f(n-1)} = \emptyset$, then the position $(f(0), f(1), f(2), ..., f(n-1))$ is a *terminal position.*

A *winning position* for player I [resp. II] is a position at which I [resp. II] has a w.s. In many of the games defined in this paper, player I wins iff a position is reached at which player II cannot make a (legal) move. We refer to such a position as a *terminal winning position* for player I.

We also want to have a Godel numbering of the formulas. So fix a Godel numbering of the formulas in the language $\{\in\}$ such that if n is the Godel number of a formula ψ, then

$$\psi = \psi(\mathrm{v_0}, \mathrm{v_1}, \mathrm{v_2}, ..., \mathrm{v_{n-1}})$$

i.e. the free variables of ψ are amongst $\mathrm{v_0}, \mathrm{v_1}, \mathrm{v_2}, ..., \mathrm{v_{n-1}}$. Let no.$\psi$ be the Godel number of ψ.

We use the following notation: $\omega = \{0, 1, 2, 3, ...\}$, $\mathbf{N} = \{1, 2, 3, ...\}$, $\alpha \in$ ON means α is an ordinal, $A \subset B$ means $A \subseteq B$ and $A \neq B$, and p_i is the i^{th} prime with $p_0 = 2$. ${}^{A}B$ is the set of functions from A into B. If $f \in {}^{A}B$, then $f|D$ is the function obtained from f by restricting its domain to $A \cap D$. If $x \in {}^{\omega}\omega$ and $i \in \omega$, let

$$\bar{x}(i) = p_0^{x(0)+1} p_1^{x(1)+1} p_2^{x(2)+1} \cdots p_{i-1}^{x(i-1)+1}.$$

If $p = (y(0), y(1), y(2), ..., y(n+m))$ is a position in a game G,

$$p_1 = (y(0); y(1); y(2); ...; y(n)),$$

and $p_2 = (y(n+1); y(n+2); y(n+3); ...; y(n+m))$, sometimes we denote p by $p_1 * p_2$ and p is called an *extension* of p_1 (even if $p = p_1$). Two positions are *compatible* if one of the positions is an extension of the other. A position p' is *consistent* with a position p if p and p' are compatible. A play $y = (y(0); y(1); y(2); ...)$ of G is *consistent* with a position p if $p = (y(0); y(1); y(2); ...; y(n-1))$ for some $n \in \omega$. A position p [respectively, a play y] is *consistent* with a sequence $\vec{u} = \langle u_i | i < \theta \rangle$ of positions if p [repectively, y] is consistent with u_i for all $i < \theta$. The moves of player I [respectively, player II] in a position p are *consistent* with a position u if there exists a position q consistent with u such that the moves of player I [respectively, player II] are identical to the moves of player I [respectively, player II] in p. The moves in p of player I [respectively, player II] are *consistent* with a sequence $\vec{u} = \langle u_i | i < \theta \rangle$ of positions if the moves in p of player I [respectively, player II] are consistent with u_i for all $i < \theta$. A position p of a game G is *consistent* with a set U of positions for G if $p \in U$. A play $y = (y(0); y(1); y(2); ...)$ of a game G is *consistent* with a set U of positions for G if $(y(0); y(1); y(2); ...; y(n-1)) \in U$ for all $n \in \omega$. If $\vec{U} = \langle U_i | i < \eta \rangle$ is a sequence such that each U_i is a set of positions of a game G, then the position p [respectively, the play y] of G is *consistent* with $\vec{U}$ whenever p [respectively, y] is consistent with U_i for all $i < \eta$. A position $p = (y(0); y(1); y(2); ...; y(n-1))$ is *consistent* with a strategy s if $y(2k) = s(y(1); y(3); y(5); ...; y(2k-1))$ when both $2k < n$ and s is a strategy for I, and $y(2k+1) = s(y(0); y(2); y(4); ...; y(2k))$ when both $2k+1 < n$ and s is a strategy for II. A play $y = (y(0); y(1); y(2); ...)$ is consistent with a strategy s if for each $n \in \omega$, $(y(0); y(1); y(2); ...; y(n-1))$ is consistent with s.

In Sections One and Two, we number definitions and theorems separately. As a result of this, we have Definition 1.3 and Theorem 1.3. We also have a lemma and a corollary to Theorem 1.3, which we call Lemma 1.3.1 and Corollary 1.3.1. We number our definitions and our results separately so that we have a nice correspondence between the number of a given theorem, the amount of determinacy mentioned in the theorem, the name of the auxiliary game defined in the proof of the theorem, and the name of the w.s. for the auxiliary game. For instance, for $\beta \in \{0,1,2\}$, the following hold:

(1) "Theorem 1.2β" states that if $L(\beta\#^1_2(0)) \models$ "every real has a sharp," then $\mathrm{Det}(\Pi^0_1, \beta * \Sigma^0_1)^*$.

(2) "Theorem $1.2\beta+1$" states that if $L(\beta\#^1_2(0))$ has indiscernibles, then $\mathrm{Det}(\Pi^0_1, \beta * \Sigma^0_1)^*_+$.

(3) If $i \in \{2\beta, 2\beta+1\}$, then an auxiliary game G^i and its w.s. s^i are defined in the proof of Theorem $1.i$.

§**0. The Preliminaries.** This Section consists of four subsections: 0.1, 0.2, 0.3, and 0.4. In Sections 0.1 and 0.2, we respectively define the models and the large cardinal hypotheses to be used in this paper. In Section 0.3, we introduce some terminology with respect to games, and in Section 0.4, we briefly review the three auxiliary games of [Du1,2].

In this section, we show that a certain amount of determinacy, $\mathrm{Det}(\Pi^0_1, k*\Sigma^0_1)^*$, follows from the Sharp Hypothesis:

$$L(0^{k\#^1_1})[\#^1_1] \models \text{"every real has a sharp."}$$

§**0.1. The Models.** In this section, we define the sharp functions $\beta\#^1_\gamma$ for $\beta, \gamma \in \omega$. Each $\beta\#^1_\gamma$ is a partial function on the reals, and for each r in the domain of $\beta\#^1_\gamma$, we sometimes write $r^{\beta\#^1_\gamma}$ for $\beta\#^1_\gamma(r)$. In Section One, we only encounter $\beta\#^1_\gamma$ for $\gamma \in \{1,2\}$. Let $\#^1_0(r) = r$ and $0\#^1_\gamma(r) = r$ for every real r. Now we give the definition of $r^{\#^1_1}$ for r a real:

Definition 0.1. Let $r \subseteq \omega$.

1.) The elements of a class C of ordinals are *indiscernibles for L(r)* iff for every formula φ, for any two increasing sequences

$$\zeta_0 < \zeta_1 < \zeta_2 < \cdots < \zeta_{n-1} \text{ and } \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{n-1}$$

of elements from C, we have

$$\text{L(r)}\models \varphi[\text{r},\zeta_0, \zeta_1, \zeta_2, ..., \zeta_{n-1}] \text{ iff } \text{L(r)}\models \varphi[\text{r},\xi_0, \xi_1, \xi_2, ..., \xi_{n-1}].$$

2.) $r^{\#^1_1}$ *exists* iff there exists an (unique) class C of indiscernibles for $L(r)$ which contains all uncountable cardinals and is closed and such that every $a \in L(r)$ is definable in $L(r)$ from r and the elements of C. The elements of C are called *Silver indiscernibles for L(r).*

3.) $r^{\#^1_1} \subseteq \omega$ is the set of Godel numbers of formulas φ such that if $\varphi = \varphi(v_0, v_1, v_2, v_3, ..., v_n)$, then for some (any) increasing sequence of $\xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{n-1}$ of Silver indiscernibles,

$$L(r) \models \varphi[\text{r},\xi_0, \xi_1, \xi_2, ..., \xi_{n-1}].$$

Sometimes we use the more standard notation $r^\#$ for $r^{\#^1_1}$. For r a real, we now define $L(r)[\#^1_1]$, the least inner model which contains r and which satisfies "every real has a sharp":

Definition 0.2. Let r be a real and let $\bar{r}$ denote the transitive closure of r.

1.) Let $\#^1_1$ be the function with domain $\{r \subseteq \omega | r^\# \text{ exists}\}$ defined by

$r \mapsto r^{\#}$. We refer to $\#^1_1$ as *the sharp function on reals* and we often denote $\#^1_1$ by $\#_1$.

2.) For any set M, we define Def(M) as the set of all $y \subseteq M$ such that for some formula φ and $x_1, x_2, x_3, ..., x_{n-1} \in M$,

$$y = \{x \in M \mid M \models \varphi[x, x_0, x_1, x_2, ..., x_{n-1}]\}.$$

3.) By transfinite recursion, we define $L_0(r)[\#_1] = \bar{r}$,

$L_\xi(r)[\#_1] = \bigcup_{\eta<\xi} L_\eta(r)[\#_1]$ if ξ is a limit ordinal, and

$L_{\xi+1}(r)[\#_1]$=Def($\mathrm{L}_\xi(r)[\#_1] \cup \{A^{\#} \mid A \in L_\xi(r)[\#_1] \cap dom[\#_1]\}$).

Finally, we let $L(r)[\#_1] = \bigcup_{\xi \in \mathrm{ON}} L_\xi(r)[\#_1]$. We define $L[\#_1]$ to be $L(\emptyset)[\#_1]$.

Now we define "the existence of $r^{\#^1_2}$," and if $r^{\#^1_2}$ exists, we define $r^{\#^1_2}$.

Definition 0.3. Let r be a real.

1.) $r^{\#^1_2}$ *exists* iff there exists an (unique) class C^1_r of indiscernibles for $L(r)[\#_1]$ which contains all uncountable cardinals and is closed and such that every $a \in L(r)[\#_1]$ is definable in $L(r)[\#_1]$ from r, $\#_1$, and elements of C^1_r.

2.) $r^{\#^1_2} \subseteq \omega$ is the set of Godel numbers of formulas φ such that if $\varphi = \varphi(v_0, v_1, v_2, v_3, ..., v_n)$, then for some (any) increasing sequence of $\xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{n-1}$ from C^1_r,

$$L(r)[\#_1] \models \varphi[r,\xi_0, \xi_1, \xi_2, ..., \xi_{n-1}].$$

If $0^{\#^1_2}$ exists, then $0^{\#^1_2}$ is a real which codes indiscernibles for $L[\#_1]$. We generalize the definitions of $L[\#^1_1]$ and $r^{\#^1_2} \subseteq \omega$ by defining $r^{\#^1_n}$ and $L[\#^1_n]$ by induction on n.

Definition 0.4. Let r be a real. Assume that we have defined $L(r)[\#^1_n]$.

1.) $r^{\#^1_{n+1}}$ *exists* iff there exists an (unique) class $\mathbf{C^n_r}$ of indiscernibles for $L(r)[\#^1_n]$ which contains all uncountable cardinals and is closed and such that every $a \in L(r)[\#^1_n]$ is definable in $L(r)[\#^1_n]$ from r, $\#^1_n$, and elements of $\mathbf{C^n_r}$.

2.) If $r^{\#^1_{n+1}}$ exists, then $r^{\#^1_{n+1}} \subseteq \omega$ is the set of Godel numbers of formulas φ such that if $\varphi = \varphi(v_0, v_1, v_2, v_3, ..., v_\ell)$, then

$$L(r)[\#^1_n] \models \varphi[r, \xi_0, \xi_1, \xi_2, ..., \xi_{\ell-1}]$$

for some (any) increasing sequence $\xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{n-1}$ from $\mathbf{C^n_r}$ (i.e. of indiscernibles for $L(r)[\#^1_n]$).

3.) Let $\#^1_{n+1}$ be the function with domain $\{r \subseteq \omega | r^{\#^1_{n+1}} \text{ exists}\}$ defined by $r \mapsto r^{\#^1_{n+1}}$. We refer to $\#^1_{n+1}$ as *the (n+1)st extended sharp function on the reals.*

4.) By transfinite recursion, we define $L_0(r)[\#^1_{n+1}] = \bar{r}$,
$L_\xi(r)[\#^1_{n+1}] = \bigcup_{\eta<\xi} L_\eta(r)[\#^1_{n+1}]$ if ξ is a limit ordinal, and
$L_{\xi+1}(r)[\#^1_{n+1}]$=Def($\mathrm{L}_\xi(r)[\#_1] \cup \{A^\# \mid A \in L_\xi(r)[\#^1_{n+1}] \cap dom[\#^1_{n+1}]\}$).
Finally, we let $L(r)[\#^1_{n+1}] = \bigcup_{\xi \in \mathrm{ON}} L_\xi(r)[\#^1_{n+1}]$. We define $L[\#^1_{n+1}]$ to be $L(\emptyset)[\#^1_{n+1}]$.

$\#_m$ [Du3] and $\#^m_n$ [Du5] generalize $\#^1_n$ (e.g. $\#_1 = \#^1_1$) and satisfy $\#_m = \#^m_1$. In [Du5], whenever $m \in \omega$ and x is of type m, we define the notion of "$x^{\#^m_n}$ exists" and if $x^{\#^m_n}$ exists, we define $x^{\#^m_n}$. By induction on k, we now define for r a real, "the existence of $r^{k\#^1_n}$" and if $r^{k\#^1_n}$ exists, we define $r^{k\#^1_n} \subseteq \omega$:

Definition 0.5. Let r be a real and recall $r^{0\#^1_n} = r$. Assume $r^{k\#^1_n} \subseteq \omega$ has been defined and exists. We say that $r^{(k+1)\#^1_n}$ *exists* whenever $(r^{k\#^1_n})^{\#^1_n}$ exists, and if $r^{(k+1)\#^1_n}$ exists, let $r^{(k+1)\#^1_n} = (r^{k\#_n})^{\#_n}$.

Notice that $0^{(k+1)\#^1_2}$ exists iff $L(0^{k\#^1_2})[\#_1]$ has an uncountable set C^1_{k+1} of indiscenibles; in this case, $0^{(k+1)\#^1_2}$ is a real which codes indiscernibles for $L(0^{k\#^1_2})[\#_1]$. $L(0^{\#^1_2})[\#_1]$ is the least inner model of

"ZFC + every real has a sharp"

which contains a real that codes indiscernibles for $L[\#_1]$. $L(0^{\#^1_2})[\#_1]$ is the least inner model of ZFC which contains sharp functions on two inner models of

"ZFC + every real has a sharp;"

it contains the two sharp functions $\#_1|L[\#_1]$ and $\#_1|L(0^{\#^1_2})[\#_1]$. Similarly, $L(0^{k\#^1_2})[\#_1]$ is the least inner model of ZFC which contains sharps functions on $k+1$ inner models of "ZFC + every real has a sharp;" it contains the $k+1$ sharp functions

$$\#_1|L[\#_1], \#_1|L(0^{\#^1_2})[\#_1], \cdots, \#_1|L(0^{k\#^1_2})[\#_1].$$

In general, $(\beta+1)\#^1_{\gamma+1}(0)$ exists iff $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma]$ has an uncountable set $C^\gamma_{\beta+1}$ of indiscernibles.

For many of the theorems in this paper, we need the following as our hypothesis: $(\beta+1)\#^1_{\gamma+1}(0)$ exists and $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma] \models$ "$r^{\#^1_\gamma}$ exists for every real r." Menachem Magidor pointed out that we only need $(\beta+1)\#^1_{\gamma+1}(0)$ exists as our hypothesis:

Lemma 0.6. If $(\beta+1)\#^1_{\gamma+1}(0)$ exists, then $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma] \models$ "$r^{\#^1_\gamma}$ exists for every real r."

Proof: Assume $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma]$ has an uncountable set C of indiscernibles and $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma] \not\models$ "$r^{\#^1_\gamma}$ exists for every real r." Then the least real in $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma]$ such that $r^{\#^1_\gamma}$ does not exist is definable in $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma]$ so that C is a set of indiscernibles for $L(r)[\#^1_{\gamma-1}]$. Thus, $r^{\#^1_\gamma}$ does exist. ∎

§**0.2. The Determinacy Hypotheses.** In this paper, we characterize

(i) the existence of $\beta\#^1_{\gamma+1}(0)$ and

(ii) $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma] \models$ "$r^{\#^1_\gamma}$ exists for every real r"

in terms of determinacy. Next we deine $\beta - \Pi^1_1$ and then we define the classes $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ and $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$, whose determinacy we relate to (i) and (ii). Eventually we show the following:

Theorem 2.4. If (ii) holds, then $\mathrm{Det}(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$; and

Theorem 2.5. If $(\beta+1)\#^1_{\gamma+1}(0)$ exists, then $\mathrm{Det}(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$.

In [Du4], we show the converse of Theorem 2.5.

Definition 0.7. Let β be an ordinal. $A \subseteq {}^\omega\omega$ is β-$\mathbf{\Pi^1_1}$ iff there exist $\mathbf{\Pi^1_1}$ sets A_α for $\alpha \leq \beta$ such that $A_\beta = \emptyset$ and

$$A = \{x \in {}^\omega\omega \mid \exists \alpha \leq \beta (\alpha \text{ odd } \& \ x \in \textstyle\bigcap_{\gamma<\alpha} A_\gamma \setminus A_\alpha)\}.$$

We also have the lightface version of this definition:

Definition 0.8. Let β be a recursive ordinal. $A \subseteq {}^\omega\omega$ is β-Π^1_1 iff $A_\alpha \subseteq {}^\omega\omega$ exist for each $\alpha \leq \beta$ with $A_\beta = \emptyset$, and there exists a recursive wellordering of a subset E of ω with order type β such that if $|n|$ is the order type of $n \in \omega$ in this wellordering, then $\{(k,x) \in E \times {}^\omega\omega | x \in A_{|k|}\} \in \Pi^1_1$ and

$$A = \{x \in {}^\omega\omega \mid \exists\alpha \leq \beta(\alpha \text{ odd } \& \ x \in \bigcap_{\gamma<\alpha} A_\gamma \setminus A_\alpha)\}.$$

In this case, we shall say that $\langle A_\alpha | \alpha \leq \beta\rangle$ *witnesses* $A \in \beta$-Π^1_1. $\langle B_\alpha | \alpha < \beta\rangle$ *witnesses* $B \in \beta$-Π^1_1 if $\langle B_\alpha | \alpha \leq \beta\rangle$ witnesses $B \in \beta$-Π^1_1 whenever we set $B_\beta = \emptyset$.

One should note that, in the definition above, if we replace "there exists a recursive wellordering" with "for every recursive wellordering," we get an equivalent definition. Furthermore, whenever we refer to such a recursive ordering, $|n|$ denotes the order type of the initial segment of $n \in \omega$ in the ordering.

Definition 0.9. If $\mathrm{R}_1, \mathrm{R}_2, \mathrm{R}_3, ..., \mathrm{R}_\mathrm{k} \subseteq ({}^\omega\omega) \times \omega$, let

$(R_1, R_2, R_3, ..., R_k) =$

$$\{(x,n) \in ({}^\omega\omega) \times \omega | \exists i {\leq} k[R_i(x,n) \ \& \ \forall j{<}i \forall m \neg R_j(x,m)]\}.$$

To check if $(x,n) \in (R_1, R_2, R_3, ..., R_k)$, we first see if $R_1(x,n)$; if not, then we check to see if $R_2(x,n)$ and $\forall m \neg R_1(x,m)$; and again if not, we eventually need for some $i \leq k$, $R_i(x,n)$ and

$$\forall m(\neg R_1(x,m) \ \& \ \neg R_2(x,m) \ \& \ \neg R_3(x,m) \ \& \ ... \ \& \ \neg \ R_{i-1}(x,m)).$$

If this never occurs, then $(x,n) \notin (R_1, R_2, R_3, ..., R_k)$.

Definition 0.10. If $\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k$ are pointclasses, we say

$$B \subseteq ({}^\omega\omega) \times \omega \text{ is } (\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)$$

iff there exist $R_1 \in \Gamma_1,\ R_2 \in \Gamma_2,\ R_3 \in \Gamma_3, ...,\ R_k \in \Gamma_k$ such that

$$B = (R_1, R_2, R_3, ..., R_k);$$

in this case, we say $\langle R_{i+1} | i < k \rangle$ *witnesses* $B \in (\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)$.

If $\Gamma_i = \Pi^0_1$ for $i = 1, 2, 3, ..., \gamma$ and $\Gamma_i = \Sigma^0_1$ for $\gamma < i \leq \gamma + \beta$, then we abbreviate $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_{\gamma+\beta})$ by $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)$. Furthermore, we write $(\gamma * \Pi^0_1)$ for $(\gamma * \Pi^0_1, 0 * \Sigma^0_1)$ and $(\beta * \Sigma^0_1)$ for $(0 * \Pi^0_1, \beta * \Sigma^0_1)$. In particular, $(1 * \Pi^0_1) = (\Pi^0_1) = \Pi^0_1$, $(1 * \Sigma^0_1) = (\Sigma^0_1) = \Sigma^0_1$, and $(0 * \Pi^0_1) = (0 * \Sigma^0_1) = \{\emptyset\}$.

Whenever we have specified a particular $\langle A_\alpha | \alpha < \omega^2 \rangle$ to witness that some set A is $\omega^2 - \Pi^1_1$, we let $A^*_{\omega \cdot n}$ be the $\omega \cdot (n+1) - \Pi^1_1$ set

$$\{x \in {}^\omega\omega | \exists \alpha < \omega \cdot (n+1)\ (\alpha \text{ odd } \&\ x \in \textstyle\bigcap_{\beta < \alpha} A_\beta \backslash A_\alpha)\};$$

and if in addition $B \subseteq ({}^\omega\omega) \times \omega$, then we let $B^*(\langle A_\alpha | \alpha < \omega^2 \rangle) =$

$$\{x \in {}^\omega\omega | \exists n [B(x,n)\ \&\ \forall m < n \neg B(x,m)\ \&\ x \in A^*_{\omega \cdot n}]\}.$$

We shall just write B^* for $B^*(\langle A_\alpha | \alpha < \omega^2 \rangle)$ whenever no confusion may arise. $A \subseteq {}^\omega\omega$ is $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ if there exist $B \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)$ and $\langle A_\alpha | \alpha < \omega^2 \rangle$ which witnesses some set is $\omega^2 - \Pi^1_1$ such that $A = B^*$; in this case, we say B and $\langle A_\alpha | \alpha < \omega^2 \rangle$ *witness* A is $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$. If in addition $B = (R_1, R_2, R_3, ..., R_{\gamma+\beta})$, then we say $R_1, R_2, R_3, ..., R_{\gamma+\beta}$ and $\langle A_\alpha | \alpha < \omega^2 \rangle$ *witness* A is $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$.

If $B \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)$ and $B(x,n)$ holds for some least n, then

$$x \in B^*(\langle A_\alpha | \alpha < \omega^2\rangle) \text{ iff } x \in A^*_{\omega\cdot n}.$$

In general, whenever $\exists n \leq m B(x,n)$, it is not necessarily true that $x \in B^*(\langle A_\alpha|\alpha < \omega^2)$ iff $x \in A^*_{\omega\cdot m}$; however, we shall now show that wlog one may assume this.

Lemma 0.11. Let $B \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)$ and $\langle A'_\alpha|\alpha < \omega^2\rangle$ witness that A is $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$. Then there exists $\langle A_\alpha|\alpha < \omega^2\rangle$ such that

(i) B and $\langle A_\alpha|\alpha < \omega^2\rangle$ witness that A is $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ and

(ii) whenever $\exists n \leq m\ B(x,n)$, $x \in A$ iff $x \in A^*_{\omega\cdot m}$.

In this case, we say that B and $\langle A_\alpha|\alpha < \omega^2\rangle$ *strongly witness* $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$.

Proof: Let $A_{\omega\cdot n} = \{x \in A'_{\omega\cdot n}|\forall m < n \neg B(x,m)\}$ and otherwise A_α is A'_α. Since $B \in \Delta^1_1$, each $\{(x,n) \in ({}^\omega\omega) \times \omega | x \in A_{|n|}\} \in \Pi^1_1$. ■

Now we define the class $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$. This class properly contains $(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$. Furthermore, for any $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$, there exists $B^* = B^*(\langle A_\alpha|\alpha < \omega^2\rangle) \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$ such that if $\exists n B(x,n)$, then $x \in A$ iff $x \in B^*$. But x may be in A even if $\forall n \neg B(x,n)$; whereas, for any $B^* \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*$, $x \notin B^*$ if $\forall n \neg B(x,n)$.