Determinacy and sequences of sharp functions on the reals

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Abstract. For several sequences $\vec{\#}$ of sharp functions on the reals, we find the determinacy strength of the following large cardinal properites:

(i) $L[\vec{\#}] \models$ "every sharp function in $\vec{\#}$ is total."

(ii) Indiscernibles for $L[\vec{\#}]$ exist.

§0. Introduction and statement of results.

This is one of a series of papers in which we investigate the determinacy strength of the existence of indiscernibles for various least inner models of sharp functions. The classes whose determinacy we consider are all within $\Delta\left(\omega^2 - \Pi^1_1\right)$ (see Definition 1.9) and the corresponding models are mice from the Dodd-Jensen Core model, the core model below a measurable. All the papers in this series are fairly self-contained. Martin's upcoming book [Ma∞] is an excellent reference for determinacy and does include much of the material in this series in Chapter 5. At the end of this section, we review some of the results concerning determinacy in the (effective) difference hierarchy on Π^1_1 sets.

In this section §0 we go directly into summarizing the results of this paper and related results. This makes §0 initially more difficult reading for those not familiar with the classes in §1.2. However, we give all the basic definitions in the Preliminaries §1 (making this paper fairly self-contained and easy to read when starting with §1). §§ 2 and 3 give the proofs of the main results of this paper and are independent of one another. If it weren't for the common preliminaries, it would make sense to publish Sections §2 and §3 separately. Logically the material in §3 should precede that in §2 but the proofs in §2 are simpler.

Let $\#^\infty$ denote the (partial) sharp function on all sets so that $\#^\infty(x)$ codes indiscernibles for $L(x)$. Let $\#^k$ be the restriction of $\#^\infty$ to objects of type k (so that $\#^1$ is the sharp function on the reals). For $k < \omega_1^{CK}$, let $\#^{(k+1)\infty}$ be the partial sharp function such that $\#^{(k+1)\infty}(x)$ codes indiscernibles for $L(x)\left[\vec{\#}^{k\infty}\right]$, where $\vec{\#}^{k\infty} = \langle \#^{(\beta+1)\infty} | \beta < k\rangle$. The determinacy strength of (the natural large cardinal properties for) $L\left[\#^k\right]$, $L\left[\#^\infty\right]$, and $L\left[\vec{\#}^{k\infty}\right]$ are all known and stated in Theorems 2 and 3 below.

The classes whose determinacy in which we are interested are $(\Gamma)^*$ and $(\Gamma)^*_+$ for various classes Γ (e.g. $\Gamma = \Pi^0_k, \Pi^1_1$). We review the definition of these classes in Definitions 1.13 and 1.14. Given $\vec{B} = \langle B_{i+1} | i < k\rangle$, let $(x,n) \in \left(\vec{B}\right)$ iff for some $i < k$,

$$B_i(x,n) \text{ and } (i,n) \text{ is lexicographically least such that } B_i(x,n).$$

$(k * \Gamma)$ is defined to be the collection of $(\langle B_{i+1} | i < k\rangle)$ for which each $B_i \in \Gamma$ and $\langle B_{i+1} | i < k\rangle$ is a Π^1_1 sequence. Let $(k*\Gamma)^* = ((k*\Gamma))^*$ and $(k*\Gamma)^*_+ = ((k*\Gamma))^*_+$.

In Theorems 1, 2, and 3 below, we indicate the correlation between our large cardinal properties for the several inner models (e.g. $L[\#^k]$, $L[\vec{\#}^{k\infty}]$) and the determinacy of the classes $(\Gamma)^*$ and $(\Gamma)^*_+$ for $\Gamma \in \{\Pi^0_k, \Sigma^0_k, \Pi^1_1, (k * \Pi^1_1) | k < \omega_1^{CK}\}$.

Theorem 1. [Du90] Let $k < \omega_1^{CK}$, let $0^{1\#} = 0^\#$, $0^{(k+1)\#} = \left(0^{k\#}\right)^\#$, and for k a limit, let $0^{k\#}$ be a real which recursively codes the sequence $\langle 0^{\beta\#} | \beta < k\rangle$. Then $0^{k\#}$ exists $\Leftrightarrow$ $\mathrm{Det}\big(k * \Sigma^0_1\big)^*_+ \Leftrightarrow \mathrm{Det}\big((k+1) * \Sigma^0_1\big)^*$.

The fact that the determinacy of the star class $((k+1) * \Sigma^0_1)^*$ and the star plus class $(k * \Sigma^0_1)^*_+$ are equivalent is somewhat of an oddity in that this rarely occurs at the higher levels within $\Delta(\omega^2 - \Pi^1_1)$.

Theorem 2.

(i) [Du92a] If $L\left[\#^1\right] \models$ "$\#^1$ is total," then $\mathrm{Det}\big(\Sigma^0_2\big)^*$ and in particular $\mathrm{Det}\big(\Pi^0_1\big)^*$.

(ii) [Du95] If $k < \omega_1^{CK}$ and $L\left[\#^k\right] \models$ "$\#^k$ is total," then $\mathrm{Det}\big(\Sigma^0_{k+1}\big)^*$ and in particular $\mathrm{Det}\big(\Pi^0_k\big)^*$.

(iii) If $\omega \le k < \omega_1^{CK}$ and $L\left[\#^k\right] \models$ "$\#^k$ is total," then $\mathrm{Det}\big(\Sigma^0_k\big)^*$ (so that $\mathrm{Det}\big(\Pi^0_{k-1}\big)^*$ whenever k is a successor ordinal).

(iv) (Martin) If $\mathrm{Det}\big(\Pi^1_1\big)^*$, then $L\left[\#^\infty\right] \models$ "$\#^\infty$ is total".

(v) (Martin) If every real has a sharp and and if $\mathrm{Det}\big(\Pi^1_1\big)^*_,$ indiscernibles for $L\left[\#^\infty\right]$ exist.

The conclusions of Theorem 2(i,ii,iii) are reductant since $(\Pi^0_k)^* = (\Sigma^0_{k+1})^*$. Also $(\Pi^0_k)^*_+ = (\Sigma^0_{k+1})^*_+$.

Theorem 3.

(i) [Du92a] Indiscernibles for $L\left[\#^1\right]$ exist $\Leftrightarrow$ $\mathrm{Det}\big(\Pi^0_1\big)^*_+ \Leftrightarrow \mathrm{Det}\big(\Sigma^0_2\big)^*_+$.

(ii) [Du95] For $k < \omega$, indiscernibles for $L\left[\#^k\right]$ exist $\Leftrightarrow$ $\mathrm{Det}\big(\Pi^0_k\big)^*_+ \Leftrightarrow \mathrm{Det}\big(\Sigma^0_{k+1}\big)^*_+$.

(iii) For $\omega \le k < \omega_1^{CK}$, indiscernibles for $L\left[\#^k\right]$ exist $\Leftrightarrow$ $\mathrm{Det}\big(\Sigma^0_k\big)^*_+$ ($\Leftrightarrow$ $\mathrm{Det}\big(\Pi^0_{k-1}\big)^*_+$ if k is a successor ordinal).

(iv) [Du7] Indiscernibles for $L\left[\#^\infty\right]$ exist $\Leftrightarrow$ $\mathrm{Det}\big(\Pi^1_1\big)^*_+$.[1]

(v) [Du7] For $k < \omega_1^{CK}$, indiscernibles for $L\left[\vec{\#}^{k\infty}\right]$ exist $\Leftrightarrow$ $\mathrm{Det}\big(k * \Pi^1_1\big)^*_+$.

Relativizations of all of the above theorems hold. To prove the results of this paper, we shall need to relativize our results to the reals.

The results of [Du95], Theorems 2(ii) and 3(ii), generalize those of [Du92a], Theorems 2(i) and 3(i), by showing that some results which hold for the sharp function on the reals also hold for sharp functions on higher type objects. In this paper, we also generalize the results of [Du92a] by showing that analogous results hold for sequences of extended sharp functions on the reals. We define several extended sharp functions on the reals, and for each such sharp function, we define several inner models of "every real has an extended sharp." Then we characterize these models in terms of determinacy.

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[1] Martin and I independently proved the ⇒ direction of 3(iv); this is what is shown in [Du7].

We also generalize Theorems 1(ii) and 2(ii, iii, iv). So as to simultaneously prove our generalizations, it is convenient to let, for r a fixed real, $\Gamma_r\left(\emptyset\right)=\emptyset$, $\Gamma_r\left(\#^k\right)=\Pi^0_k(r)$ for $1\le k<\omega$, $\Gamma_r\left(\#^k\right)=\Sigma^0_k(r)$ for $\omega\le k<\omega_1^{CK}(r)$, and $\Gamma_r\left(\vec{\#}^{k\infty}\right)=\left(k*\Pi^1_1\right)(r)$ for $k\le\omega_1^{CK}(r)$. Then we can restate the corresponding relativizations of Theorem 3 to a real r as follows:

(iv) Let r be a real, $k<\omega_1^{CK}(r)$, and $\vec{\#}\in\left\{\#^k,\vec{\#}^{k\infty}\right\}$. Then indiscernibles for $L\left(r\right)\left[\vec{\#}\right]$ exist iff $\mathrm{Det}\left(\Gamma_r(\vec{\#})\right)^*_+(r)$. $\Big($The relatived class $\left(\Gamma\right)^*_+(r)$ is defined in Definition 1.18.$\Big)$

For certain sequences $\vec{\#}$ of sharp functions (e.g. $\emptyset$, $\#^k$, $\#^{k\infty}$), we set $\vec{\#}_0=\vec{\#}$ and define the partial sharp function $\#_{\gamma+1}=\mathcal{S}_1^{\gamma+1}\left(\vec{\#}\right)$ on the reals so that $\#_{\gamma+1}\left(r\right)$ codes indiscernibles for $L\left(r\right)\left[\vec{\#}_\gamma\right]$, where $\vec{\#}_\gamma=\langle\#_{\beta+1}|\beta<\gamma\rangle$. (We do not define $\#_\gamma=\mathcal{S}^\gamma(\vec{\#})$ for γ a limit.) Let

$$\#^1_{\gamma+1}=\mathcal{S}_1^{\gamma+1}\left(\emptyset\right),\ \#^{k,1}_{\gamma+1}=\mathcal{S}_1^{\gamma+1}\left(\#^k\right),\text{ and }\#^{k\infty,1}_{\gamma+1}=\mathcal{S}_1^{\gamma+1}\left(\vec{\#}^{k\infty}\right).$$

In this paper, we show:

Theorem 4. Let r be a real, $k,\gamma<\omega_1^{CK}$, and $\vec{\#}_0\in\left\{\emptyset,\#^k,\vec{\#}^{k\infty}\right\}$.

(a) Assuming $\vec{\#}_\gamma\neq\vec{\#}^{k\infty}$ (i.e. $\gamma\ge1$ for the case $\vec{\#}_0=\vec{\#}^{k\infty}$), if $L\left(r\right)\left[\vec{\#}_\gamma\right]\models\forall\beta<\gamma$ "$\#_{\beta+1}$ is total," then every $\left(\Gamma(\vec{\#}_0),\gamma*\Pi^0_1\right)^*(r)$ game is determined and has a w.s. in $L\left(r\right)\left[\vec{\#}_\gamma\right]$.

(b) $\#_{\gamma+1}\left(r\right)$ exists i.e. indiscernibles for $L\left(r\right)\left[\vec{\#}_\gamma\right]$ exist iff every $\left(\Gamma(\vec{\#}_0),\gamma*\Pi^0_1\right)^*_+(r)$ game is determined, in which case each such game has a w.s. in $L\left(\#_{\gamma+1}\left(r\right)\right)$.

Remark. We need $\gamma\ge1$ when $\vec{\#}_0=\vec{\#}^{k\infty}$ in (a) above since the converse of 2(iv) is false. For the case in which γ is a successor ordinal, we can replace $L\left(r\right)\left[\vec{\#}_\gamma\right]\models\forall\beta<\gamma$ "$\#_{\beta+1}$ is total" in (a) above with $L\left(r\right)\left[\vec{\#}_\gamma\right]\models$ "$\#_\gamma$ is total". When γ is a limit (and $\neq0$), $\#_\gamma$ is undefined.

Theorem 4 holds with much more general sequences $\vec{\#}_0$. Now we generate such sequences, with $\vec{\#}_0\in\{\#^\infty,\#^k|k<\omega_1^{CK}\}$ as our starting points (the reader can instead take $\vec{\#}_0=\emptyset$ as the start). Given a sequence $\vec{\#}$ of sharp functions, let $\mathcal{S}_\infty(\vec{\#})$ be the sharp function for which $(\mathcal{S}_\infty(\vec{\#}))(x)$ codes indiscernibles for the model $L(x)[\vec{\#}]$. Let $\mathcal{S}_k(\vec{\#})$ be the restriction of $\mathcal{S}_\infty(\vec{\#})$ to objects of type k. Fixing $k\in ON\cup\{\infty\}$, we inductively define both the sequence $\vec{\mathcal{S}}_k^\alpha(\vec{\#})$ extending $\vec{\#}$ and the sharp function $\mathcal{S}_k^{\alpha+1}(\vec{\#})$ as follows:

- $\mathcal{S}_k^{\alpha+1}(\vec{\#})=_{df}\mathcal{S}_k(\vec{\mathcal{S}}_k^\alpha(\vec{\#}))$ and
- $\vec{\mathcal{S}}_k^\alpha(\vec{\#})=_{df}\vec{\#}^\frown\langle\mathcal{S}_k^{\beta+1}(\vec{\#})|\beta<\alpha\rangle$.

(For α an infinite limit ordinal, $\mathcal{S}_k^\alpha(\vec{\#})$ is undefined whereas $\vec{\mathcal{S}}_k^\alpha$ is defined.) Note that:

- $\vec{\mathcal{S}}_k^0(\vec{\#})=\vec{\#}$,
- $\vec{\mathcal{S}}_k^{\alpha+1}(\vec{\#})$ extends $\vec{\mathcal{S}}_k^\alpha(\vec{\#})$ by the sharp function $\mathcal{S}_k(\vec{\mathcal{S}}^\alpha(\vec{\#}))$, and
- for α an infinite limit ordinal, $\vec{\mathcal{S}}^\alpha(\vec{\#})$ is the common extension of the $\vec{\mathcal{S}}^\beta(\vec{\#})$ for which $\beta<\alpha$.

Since $\mathcal{S}_k(\mathcal{S}_\ell(\vec{\#}))=\mathcal{S}_k(\vec{\#})$ when $\ell<k$, the sequences of interest are:

$$\vec{\#}=\vec{\mathcal{S}}_{k(\ell)}^{\alpha(\ell)}\vec{\mathcal{S}}_{k(\ell-1)}^{\alpha(\ell-1)}\vec{\mathcal{S}}_{k(\ell-2)}^{\alpha(\ell-2)}\cdots\vec{\mathcal{S}}_{k(1)}^{\alpha(1)}\vec{\mathcal{S}}_{\infty}^{\alpha(0)}(\emptyset),$$

where $k(1) > k(2) > k(3) > \cdots > k(\ell)$. We shall consider such $\vec{\#}$ for which the $k(i)$'s and the $\alpha(i)$'s are $< \omega_1^{CK}$.

For such $\vec{\#}$, we now define the class $\Gamma(\vec{\#})$. Given $\vec{B} = \langle B_i | i < k\rangle$ for which each $B_i \subseteq ({}^{\omega}\omega) \times \omega$, let

$$(\vec{B})(x,n) \Leftrightarrow_{df} \exists i < k \exists m B_i(x,m) \text{ and for the lexicographically least such } (i,m) \text{ we have } m = n.$$

Given $\vec{\Gamma} = \langle \Gamma_i | i < k\rangle$, let $(\vec{\Gamma})$ be the collection of all $(\langle B_i | i < k\rangle)$ for which each $B_i \in \Gamma_i$. Let $(\alpha(1) * \Gamma_{k(1)}, \alpha(2) * \Gamma_{k(2)}, ..., \alpha(\ell) * \Gamma_{k(\ell)})$ denote $(\langle \tilde{\Gamma}_i | i < \alpha(1) + \alpha(2) + \alpha(3) + \cdots + \alpha(\ell)\rangle)$, where

$$\tilde{\Gamma}_i =_{df} \begin{cases} \Gamma_{k(1)} & \text{for } i \in [0, \alpha(1)) \\ \Gamma_{k(j)} & \text{for } i \in [\alpha(j-1), \alpha(j)) \text{ and } 2 \leq j \leq \ell. \end{cases}$$

Let

$$\Gamma\left(\vec{\mathcal{S}}_{k(\ell)}^{\alpha(\ell)} \vec{\mathcal{S}}_{k(\ell-1)}^{\alpha(\ell-1)} \vec{\mathcal{S}}_{k(\ell-2)}^{\alpha(\ell-2)} \cdots \vec{\mathcal{S}}_{k(1)}^{\alpha(1)} \vec{\mathcal{S}}_{\infty}^{\alpha(0)}(\emptyset)\right) =_{df} \left(\alpha(0) * \Pi_1^1, \alpha(1) * \Gamma(\#_{k(1)}), \alpha(2) * \Gamma(\#_{k(2)}), ..., \alpha(\ell) * \Gamma(\#_{k(\ell)})\right)$$

if $\omega_1^{CK} > k(1) > k(2) > k(3) > \cdots > k(\ell)$ and the $\alpha(i)$'s are $< \omega_1^{CK}$.

Theorem 5. Theorem 4 holds with $\vec{\#}_0 = \vec{\mathcal{S}}_{k(\ell)}^{\alpha(\ell)} \vec{\mathcal{S}}_{k(\ell-1)}^{\alpha(\ell-1)} \vec{\mathcal{S}}_{k(\ell-2)}^{\alpha(\ell-2)} \cdots \vec{\mathcal{S}}_{k(1)}^{\alpha(1)} \vec{\mathcal{S}}_{\infty}^{\alpha(0)}(\emptyset)$, if $\omega_1^{CK} > k(1) > k(2) > k(3) > \cdots > k(\ell)$ and the $\alpha(i)$'s are $< \omega_1^{CK}$.

Section 3 is devoted to proving Theorems 4 and 5. We present the material in Section 3 so that the reader can interpret the proofs there as referring to some very specific value for $\vec{\#}_0$ (as in Theorem 4) or to the more general case given in Theorem 5. For $\beta < \omega_1^{CK}$ and $\#$ a sharp function, let $1\# = \#$, $((\beta+1)\#)(r) = \#(\beta\#(r))$, and for β a limit, let $\beta\#(r)$ recursively code the sequence $\left\langle \hat{\beta}\#(r) | \hat{\beta} < \beta \right\rangle$ of reals. We generalize Theorem 5 by showing the following:

Theorem 6. Let $\beta < \omega_1^{CK}$. Let r, k, γ be as in Theorem 4, and let $\vec{\#}_0$ be as in either Theorem 4 or 5.

(c) Assuming $\vec{\#}_\gamma$ does not equal $\vec{\#}^{\alpha\infty}$ for any $\alpha \geq 1$ (i.e. $\gamma \geq 1$ when $\vec{\#}_0 = \vec{\#}^{k\infty}$), if $L(\beta\#_{\gamma+1}(r))\left[\vec{\#}_\gamma\right] \models \forall\hat{\gamma} < \gamma$ "$\#_{\hat{\gamma}+1}$ is total," then every $\left(\Gamma(\vec{\#}_0), \gamma * \Pi_1^0, \beta * \Sigma_1^0\right)^*(r)$ game is determined and has a w.s. in $L(\beta\#_{\gamma+1}(r))\left[\vec{\#}_\gamma\right]$.

(d) $(\beta+1)\#_{\gamma+1}(r)$ exists (i.e. indiscernibles for $L(\beta\#_{\gamma+1}(r))\left[\vec{\#}_\gamma\right]$ exist) iff every $\left(\Gamma(\vec{\#}_0), \gamma * \Pi_1^0, \beta * \Sigma_1^0\right)_+^*(r)$ game has a w.s. in $L\Big((\beta+1)\#_{\gamma+1}(r)\Big)$.

Theorem 5 is the case of Theorem 6 where $\beta = 0$. Theorem 6 is proved by induction on β with Theorem 5 as our base step. In this setting, our proof of Theorem 6 requires that we consider only one pair of Borel auxiliary moves (see Definitions 2.6 and 2.10) in the corresponding auxiliary game, whereas some plays of the auxiliary games in our proof of Theorems 4 and 5 require ω many such pairs. We therefore prove (the more general) Theorem 6 in Section 2 and Theorems 4 and 5 in Section 3. As with Section 3 and Theorems 4 and 5, we present the material in Section 2 so that the reader can interpret the proofs there as referring to some very specific value for $\vec{\#}_0$ or to the general case given in Theorem 6. Section 1 is the Preliminaries in which we provide the definitions of the classes, the models, etc.

The classes whose determinacy we discuss in Theorems 5 and 6 form a rich hierarchy of classes between each $\left(\alpha(0) * \Pi^1_1\right)^*_+$ and $\left((\alpha(0)+1) * \Pi^1_1\right)^*$. Let $\mathcal{F}$ be the collection of $f : \omega_1^{CK} + 1 \to \omega_1^{CK}$ which take the value 0 on all but a finite set and $f\left(0\right) = 0$. If $f \in \mathcal{F}$ and

$$\omega_1^{CK} > k\left(m\right) > k\left(m-1\right) > k\left(m-2\right) > \cdots > k\left(\ell\right) \geq \omega > k\left(\ell-1\right) > k\left(\ell-2\right) > \cdots > k\left(1\right) \geq 1$$

is a descending sequence of ordinals such that $\left\{i < \omega_1^{CK} | f\left(i\right) \neq 0\right\} \subseteq \{k(j) | j \leq m\}$, we define the class $\Pi\left(f\right)$ and the sequence $\vec{\#}_f$ of sharp functions as follows:

$$\vec{\#}_f =_{df} \vec{\mathcal{S}}^{f(k(\ell))}_{k(\ell)} \vec{\mathcal{S}}^{f(k(\ell-1))}_{k(\ell-1)} \vec{\mathcal{S}}^{f(k(\ell-2))}_{k(\ell-2)} \cdots \vec{\mathcal{S}}^{f(k(1))}_{k(1)} \vec{\mathcal{S}}^{f(k(0))}_{\infty}(\emptyset)$$

$$\begin{aligned}\Pi\left(f\right) &=_{df} \Gamma(\vec{\#}_f)\\ &= \Big(f\left(\omega_1^{CK}\right) * \Pi^1_1, f\left(\alpha\left(m\right)\right) * \Sigma^0_{\alpha(m)}, f\left(\alpha\left(m-1\right)\right) * \Sigma^0_{\alpha(m-1)}, \cdots, f\left(\alpha\left(\ell\right)\right) * \Sigma^0_{\alpha(\ell)},\\ &\qquad f\left(\alpha\left(\ell-1\right)\right) * \Pi^0_{\alpha(\ell-1)}, f\left(\alpha\left(\ell-2\right)\right) * \Pi^0_{\alpha(\ell-2)}, \cdots, f\left(\alpha(1)\right) * \Pi^0_{\alpha(1)}\Big).\end{aligned}$$

Clearly all the classes of Theorems 4, 5, and 6 have the form $\left(\Pi\left(f\right), \beta * \Sigma^0_1\right)^*$ and $\left(\Pi(f), \beta * \Sigma^0_1\right)^*_+$ for some $f \in \mathcal{F}$. Moreover, these are the natural $(\cdots)^*$ and $(\cdots)^*_+$ classes to consider since $\left(\cdots, \tilde{\Gamma}, \Gamma, \cdots\right) = (\cdots, \Gamma, \cdots)$ when $\tilde{\Gamma} \subseteq \Delta(\Gamma)$ and Γ has sufficient closure properties e.g. $\left(\Pi^0_\alpha, \Pi^0_\beta\right)^* = \left(\Pi^0_\beta\right)^*$ and $\left(\Pi^0_\alpha, \Pi^0_\beta\right)^*_+ = \left(\Pi^0_\beta\right)^*_+$ when $\alpha < \beta$.

$$\left(\Pi\left(g\right), \beta * \Sigma^0_1\right)^* \subset \left(\Pi\left(g\right), \beta * \Sigma^0_1\right)^*_+ \subset \left(\Pi\left(f\right), \hat{\beta} * \Sigma^0_1\right)^* \subset \left(\Pi\left(f\right), \hat{\beta} * \Sigma^0_1\right)^*_+$$

iff either $\left(f = g \text{ and } \beta < \hat{\beta}\right)$ or $\exists \xi < \omega_1^{CK} \left[g\left(\xi\right) < f\left(\xi\right) \text{ and } f\left(\zeta\right) = g\left(\zeta\right) \text{ for } \zeta > \xi\right]$.[2]

The $\vec{\#}_f$ and $\Pi(f)$ notation may be familiar since none of my pre-1996 presentations of the type of material given here used $\mathcal{S}_k$ or $\mathcal{S}_\infty$ but instead $\Pi(f)$ was defined directly and $\vec{\#}_f$ was defined inductively using the following partial order $<^{\mathcal{F}}$:

$$g <^{\mathcal{F}} f \Leftrightarrow_{df} \exists \xi \leq \omega_1^{CK} \Big(g\left(\xi\right) < f(\xi),\ g\left(\zeta\right) = f\left(\zeta\right) \text{ for } \zeta > \xi, \text{ and } g\left(\zeta\right) = 0 \text{ for } \zeta < \xi\Big).$$

Let $t\left(f\right)$ be the least ξ such that $f\left(\xi\right) > 0$.[3] If $f\left(t\left(f\right)\right) = 1$ and f is 0 otherwise, then

$$\vec{\#}_f = \#_f =_{df} \begin{cases} \#^\infty & \text{if } t\left(f\right) = \omega_1^{CK}, \\ \#^{t(f)} & \text{if } t\left(f\right) < \omega_1^{CK}. \end{cases}$$

If $f\left(t\left(f\right)\right)$ is a successor ordinal, then

(v) $$\left(f-1\right)\left(\xi\right) =_{df} \begin{cases} f\left(\xi\right) - 1 & \text{if } \xi = t\left(f\right), \\ f\left(\xi\right) & \text{otherwise.} \end{cases}$$

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[2] Note that the value of $f(\zeta)$ and $g(\zeta)$ for $\zeta < \xi$ play no role here.

[3] t in $t(f)$ is for type – see (vi).

(vi) $\#_f$ is a (partial) sharp function on objects of type $t(f)$.

(vii) $\#_f(x)$ codes indiscernibles for $L(x)\left[\vec{\#}_{f-1}\right]$.

(viii) $\vec{\#}_f = \vec{\#}_{f-1} {}^\frown \#_f$.

If $f(t(\xi))$ is a limit ordinal, then $\vec{\#}_f$ is the common extension of all $\vec{\#}_g$ for which $g <^{\mathcal{F}} f$, i.e. $\vec{\#}_f =_{df} \langle \#_g | g <^{\mathcal{F}} f$ and $g(t(g))$ is a successor ordinal $\rangle$; in this case we do not define $\#_f$.

Let $\#_{f+1}(r)$ code indiscernibles for the model $L(r)[\vec{\#}_f]$.[4] Since

$$\vec{\#}_f =_{df} \vec{\mathcal{S}}^{f(k(\ell))}_{k(\ell)} \vec{\mathcal{S}}^{f(k(\ell-1))}_{k(\ell-1)} \vec{\mathcal{S}}^{f(k(\ell-2))}_{k(\ell-2)} \cdots \vec{\mathcal{S}}^{f(k(1))}_{k(1)} \vec{\mathcal{S}}^{f(k(0))}_{\infty}(\emptyset) \text{ and } \Pi(f) = \Gamma(\vec{\#}_f),$$

we may restate Theorem 6 as follows:

Theorem 6′. Let r be a real, $f \in \mathcal{F}$, and $\beta < \omega_1^{CK}$.

(c) If $t(f) < \omega_1^{CK}$ and $L(\beta\#_{f+1}(r))\left[\vec{\#}_f\right] \models$ " every sharp function in the sequence $\vec{\#}_f$ is total," then every $\left(\Pi(f), \beta * \Sigma^0_1\right)^*(r)$ game is determined and has a w.s. in $L(\beta\#_{f+1}(r))\left[\vec{\#}_f\right]$.

(d) $(\beta+1)\#_{f+1}(r)$ exists (i.e. indiscernibles for $L(\beta\#_{f+1}(r))\left[\vec{\#}_f\right]$ exist) iff every $\left(\Pi(f), \beta * \Sigma^0_1\right)^*_+(r)$ game is determined. In this case, each such game has a w.s. in $L((\beta+1)\#_{f+1}(r))$.

Theorems 6 and 6′ holds for r being of higher type than a real.

The proofs in this paper show Theorems 6 and 6′ under the assumption that a boldface analogue of Theorem 6(d) holds for the case $t(f) > 1$ and $\beta = 0$.

Fix f such that $t(f) > 1$. Recall $\Gamma\left(\vec{\#}_f\right) = \Pi(f)$. If $\vec{\#}_0 = \vec{\#}_g$ and $\vec{\#}_\gamma = \vec{\#}_f$, where $\gamma = f(1)$ and $g(\xi) = f(\xi)$ for $\xi > 1$, then Theorem 6 is exactly Theorem 6′ with $\vec{\#}_0 = \vec{\#}_g$; also in 4(a) and 6(c), we need $\gamma \geq 1$ when $t(g) = \omega_1^{CK}$, since $t(g) = \omega_1^{CK}$ exactly when $\vec{\#}_g = \vec{\#}^{g(\omega_1^{CK})\infty}$.

To keep matters simple, we stated the results with $\beta, \gamma < \omega_1^{CK}$ in this section. In the other sections, we state and prove the (slightly more general) results for $\beta, \gamma < \omega_1^{CK}(r)$, where r is the real fixed at the beginning of each of the theorems.

This paper was initially to be a rewrite of [Du5]. [Du5] proves Theorems 4 and 5 for the simpliest interesting case, when $\#_{\gamma+1} = \#^1_{\gamma+1} = \mathcal{S}_1^{\gamma+1}(\emptyset)$ and γ and β are both finite. Our proofs there are based on my attempt to define the "underlying" open games, whereas our proofs in this paper heavily rely on an induction hypothesis assuming the existence of winning strategies for $\left(\Gamma, (<\gamma) * \Pi^0_1\right)^*_+(r)$ games. Defining such "underlying" open games is a more simplistic approach but is quite messy since it involves several

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[4] $\#_{f+1}(r)$ is to be defined so as to have the same type as r. In this paper, we shall be interested in $\#_{f+1}(r)$ for r a real, and so in this case, one may take $f+1$ to the function which takes the same values as f except $(f+1)(1) = f(1)+1$. It is more natural to define $f+1$ to be the same as f except $(f+1)(t(f)) = f(t(f))+1$, and realize we are interested here in $\#_{f+1}$ restricted to the reals. In this paper, we choose to avoid defining $f+1$.

"layers" of sequences of Borel auxiliary moves and ordinal auxiliary moves (whereas the proofs in this paper require at most one such layer). The presentation in [Du5] is very long and so it became unclear where to publish it.

The proofs of Theorems 4 and 5 for the separate cases $\vec{\#}_0 \in \{\emptyset, \#^k, \vec{\#}^{k\infty}, \vec{\#}_f\}$ are easier to unify with the presentation here, and possibly it may be clearer how to adopt the proofs presented here to obtain similar determinacy results for other models closed under sharp functions and for the existence of indiscernibles of such models, but involving the determinacy of classes beyond those presented here.

As noted at the beginning of this section, this is one of a series of papers in which we investigate determinacy within $\Delta(\omega^2 - \Pi^1_1)$. We end this section citing other determinacy results in the (effective) difference hierarchy on Π^1_1 sets. Martin's book [Ma∞] on determinacy is the best reference. Determinacy in the difference hierarchy of Π^1_1 sets is presented in Chapter 5 of [Ma∞] and in Section 31 of [Ka94].

Theorems 7 and 8 below characterize the determinacy of $\alpha - \Pi^1_1$ games for $\alpha < \omega_1^{CK}$. Theorem 7 is a well-known result from the 1970's and Theorem 8 is a much more recent result.

Theorem 7. [Ma∞] Suppose $\omega^2\gamma < \alpha < \omega^2(\gamma+1) < \omega_1^{CK}$. Then the determinacy of $\alpha - \Pi^1_1$ games is equivalent to the existence of indiscernibles for the least inner model of γ measurable cardinals.

Theorem 8. [We96] Let $\alpha > 0$ be a recursive ordinal. The following are equivalent:

(i) $\mathrm{Det}(\omega^2\alpha - \Pi^1_1)$.

(ii) There exists a clever α-mouse.

(iii) (Indiscernibility Property) There is a class C, closed and unbounded beneath every uncountable cardinal, so that for any two $\omega\alpha$ ascending sequences $\vec{c}$, $\vec{d}$ from C, setting $\mathcal{A}[\vec{c}] = \langle L_\gamma[\vec{c}], \in, \vec{c}\rangle$ equal to the smallest transitive admissible set containing the function $\vec{c}$, we have:

$$\mathcal{A}[\vec{c}] \models \sigma \Leftrightarrow \mathcal{A}[\vec{d}] \models \sigma$$

for any sentence σ of the form $\exists x_0 < c_{i_0} \exists x_1 < c_{i_1} \cdots \exists x_j < c_{i_j} \Psi$ where $j < \omega$ and Ψ is a boolean combination of Π_1 and Σ_1 formulae in the language $\mathcal{L}_{\{\dot{\in}, \dot{\vec{c}}\}}$.

Initially Solovay, and then Friedman, Martin, and Solovay proved results establishing that the existence of L_μ follows from "low" levels of projective determinacy (e.g. Σ^1_3, Δ^1_2, $(\omega^2+1) - \Pi^1_1$). By the middle 1970s, Martin showed that

(ix) the determinacy of $\omega^2 - \Pi^1_1$ games follows from the existence of L_μ.

Martin also showed that (for γ recursive) the determinacy of $< \omega^2\gamma - \Pi^1_1 =_{df} \bigcup_{\beta < \omega^2\gamma} \beta - \Pi^1_1$ games is equivalent to the existence of indiscernibles for the least inner model of γ measurables (see Theorem 7). In particular, Martin showed, by the early 1970's, that the determinacy of $< \omega^2 - \Pi^1_1$ games and the existence of $0^\#$ are equivalent. In 1975, Martin showed that Det$(3-\Pi^1_1)$ yields the existence of $0^\#$, and soon after, Harrington [Ha78] reduced this to Det(Π^1_1). Thus, the existence of $0^\#$ is equivalent to Det$(\alpha-\Pi^1_1)$ for some (all) $\alpha\in(0,\omega^2)$.

It follows from Theorem 7 that if $\omega^2\gamma<\alpha<\beta<\omega^2(\gamma+1)<\omega_1^{CK}$, then the determinacy of $\alpha-\Pi^1_1$ and $\beta-\Pi^1_1$ games are equivalent. The only known proof of this equivalence goes through the existence of indiscernibles for the model given in Theorem 7. This is the current situation even for the special cases of these determinacy equivalences (e.g. $Det(\Pi^1_1)\Leftrightarrow Det(2-\Pi^1_1)$).

Martin also established Borel Determinacy [Ma75] and later generalized this, establishing quasi-Borel determinacy [Ma90]. Quasi-Borel determinacy has important applications. Martin applied quasi-Borel determinacy in combination with his proof of (ix) to obtain for $\gamma<\omega_1^{CK}$, the determinacy of $\Delta(\omega^2\gamma-\Pi^1_1)$ from the least inner model of γ measurables. By Theorem 7, this is an optimal result.

The proof of Theorem 6 also uses quasi-Borel determinacy. Martin and the author [Du7] independently used quasi-Borel determinacy to prove the following special case of Theorem 6: the determinacy of $(\Pi^1_1)^*_+$ games follows from the existence of indiscernibles for $L[\#^\infty]$. In [Du7], we generalize this to prove the getting determinacy direction of 6(d) for the case $\vec{\#}_\gamma=\vec{\mathcal{S}}^\gamma_\infty(\emptyset)=\vec{\#}^{\gamma\infty}$. Assuming this result, we carry out the proof of this paper (using the proof of Borel determinacy), and so our use of quasi-Borel determinacy is hidden.

From (ix), it has long been known that the Core Model below a measurable contains winning strategies for $\omega^2-\Pi^1_1$. In 1985, Martin discovered an approximate characterization of Det$(\omega^2-\Pi^1_1)$ in terms of an indiscernibility property similar to 8(iii) above. In the 1990's, Philip Welch (see page 445 of [Ka94]) found a mouse-equivalence for Det$(\omega^2-\boldsymbol{\Pi}^1_1)$. By 1994, Philip proved Theorem 8, which characterizes the determinacy of $\omega^2-\Pi^1_1$ games as being equivalent to the existence of a clever mouse. He [We2] then characterized the determinacy of $\Delta(\omega^2-\Pi^1_1)$ games in terms of mice.

Whether a mouse is clever depends on its "Q-structure" (see [We96]). Theorem 3.10 of [Du7] shows that $(<\omega_1^{CK}*\Pi^1_1)^*_+ =_{df} \bigcup_{k<\omega_1^{CK}}(k*\Pi^1_1)^*_+\subseteq\Delta(\omega^2-\Pi^1_1)$. From Welch's analysis of Q-structures [We96], it is likely that $(<\omega_1^{CK}*\Pi^1_1)^*_+$ lies near the bottom of $\Delta(\omega^2-\Pi^1_1)$. It is clear how to characterize the determinacy of classes slightly above $(<\omega_1^{CK}*\Pi^1_1)^*_+$ but we do not yet have characterizations of determinacy of some sufficiently absolute classes whose union equals $\Delta(\omega^2-\Pi^1_1)$.