In the proof of Theorem 2.0, we integrated the w.s. $s_0^2 \in L[\#_2^1]$ for G_0^2 and obtained a w.s. for a $(2*\Pi_1^0)^*$ game. That proof is analogous to the proof of Theorem 1.0. In the proof of Theorem 1.0, we show how to obtain a w.s. for a $(\Pi_1^0)^*$ game by integrating a w.s. s_0^1 for G_0^1. Analogous to the proofs of Theorems 1.0 and 2.0, we could integrate a w.s. $s_0^3 \in L[\#_3^1]$ for a game G_0^3 and obtain a w.s. for a $(3*\Pi_1^0)^*$ game. Instead, we next show the more general result: We show in Theorem 2.2 how to obtain winning strategies for $(\gamma*\Pi_1^0)^*$ games whenever $L[\#_\gamma^1] \models$ "$r^{\#_\gamma^1}$ exists for every real r." As expected, the proof of this result is similar to the proof of Theorem 2.0. One significant difference in the two proofs is the auxiliary game G_0^2 of Theorem 2.0 has at most two sequences of Borel auxiliary moves–namely, $\langle U_k | \forall i < k\ \hat{u}_i = 0\rangle$ and $\langle T_n | \forall i < n\ \hat{t}_i = 0\rangle$–whereas, the auxiliary game G_0^γ of Theorem 2.2 can have as many as γ sequences of Borel auxiliary moves. However, the integration of the w.s. s_0^γ for G_0^γ with respect to each of these sequences is analogous to the integration of s_0^2 with respect to either sequence of Borel auxiliary moves in G_0^2.

The proof of Theorem 2.2 is part of an inductive proof. We show that Theorems 2.2 and 2.3 and Corollaries 2.2.1 and 2.3.1 by induction on γ. For the Base Step, use either Theorems 1.0 and 1.1 or Theorems 2.0 and 2.1 and Corollaries 2.0.1 and 2.1.1. We assume the following during the proofs of Theorems 2.2 and 2.3 and Corollaries 2.2.1 and 2.3.1:

Induction Hypothesis. Suppose $B_{\gamma-1}, B_{\gamma-2}, B_{\gamma-3}, ..., B_1 \in \Pi^0_1$, $\langle A_\alpha|\alpha < \omega^2\rangle$, and $m \in \omega$ witness that A is $((\gamma-1) * \Pi^0_1)^*_+$. Also, suppose $\vec{U}$ is a finite sequence of I-imposed subgames of G_A and $\vec{u}$ is a sequence of legal positions in G_A of odd length. Assume $\#^1_\gamma(\vec{U})$ exists and $s^{\gamma-1}_0 \in L(\vec{U})[\#^1_{\gamma-1}]$ is a w.s. for the $G^{\gamma-1}_0(\vec{U};\vec{u})$ auxiliary game determined by $B_{\gamma-1}, B_{\gamma-2}, B_{\gamma-3}, ..., B_1$, $\langle A_\alpha|\alpha < \omega^2\rangle$, and m. Then $s^{\gamma-1}_0$ can be integrated so as to obtain a w.s. $s \in L(\#^1_\gamma(\vec{U}))$ for $A(\vec{U};\vec{u})$ such that the following hold:

i.) s is a w.s. of the player for whom $s^{\gamma-1}_0$ is a w.s.

ii.) If s is a w.s. for I, p is a position consistent with s, and the moves in p of player II are consistent with $\vec{u}$, then $p \in \vec{U}$. Therefore, if s is a w.s. for I and x is consistent with s, then $x \in A(\vec{u})$.

iii.) If p is a position of the game $A(\vec{U})$ such that the moves in p of the player for whom s is not a w.s. are consistent with $\vec{u}$, then p is consistent with $\vec{u}$.

Recall that $\#^1_\gamma$ is the partial sharp function whose domain is a set of reals and which sends r to $r^{\#^1_\gamma}$. Now we characterize

$$L[\#^1_\gamma] \models \text{“}r^{\#^1_\gamma} \text{ exists for every real } r\text{”}$$

in terms of determinacy.

Theorem 2.2. If $L[\#^1_\gamma] \models$“$r^{\#^1_\gamma}$ exists for every real r,” then $\mathrm{Det}(\gamma * \Pi^0_1)^*$.

Proof: Assume $L[\#^1_\gamma] \models$“$r^{\#^1_\gamma}$ exists for every real r.” Let

$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi^0_1$ and $\langle \mathrm{A}_\alpha \mid \alpha < \omega^2\rangle$ strongly witness $\mathrm{A} \in (\gamma * \Pi^0_1)^*$.

Wlog

$$B_\gamma \subseteq B_{\gamma-1} \subseteq B_{\gamma-2} \subseteq \cdots \subseteq B_1.$$

Also, for $1 \le i \le \gamma$, there exist R_i in Δ^0_1 such that

i.) $B_i(x,n) \leftrightarrow \forall k R_i(\bar{x}(k),n)$,

and

ii.) if $\neg R_i(\bar{x}(k),n)$ and $\forall j < k R_i(\bar{x}(j),n)$, then k is odd.

We show that G_A has a w.s. s. Condition (ii) helps to simplify the proof.

We describe an open game G^γ_0 which has a w.s. $s^\gamma_0 \in L[\#^1_\gamma]$. We integrate s^γ_0 to get the w.s. $s \in L[\#^1_\gamma]$ for G_A. The moves of G^γ_0 are the integer moves $x(i)$ of G_A as well as two types of auxiliary moves: Borel auxiliary moves and ordinal auxiliary moves.

G^γ_0 contains a sequence

$$\langle (T^1_k; \langle \hat{t}^1_k, t^1_k \rangle) | k \in \omega \ \&\ \forall j < k\ \hat{t}^1_j = 0 \rangle$$

of Borel auxiliary moves, which is related to the Π^0_1 set B_1 via R_1. Player I may only play $T^1_k \in L[\#^1_\gamma]$. If II plays $\hat{t}^1_k = 0$ for all $k \in \omega$, then the play of G^γ_0 is

$$\begin{array}{lcccccccccccc} \text{I} & T^1_0 & & x(0) & & T^1_1 & & x(2) & & \cdots & T^1_{n-1} & & x(2n-2) & & \cdots \\ \text{II} & & \langle 0, t^1_0 \rangle & & x(1) & & \langle 0, t^1_1 \rangle & & x(3) & & & \langle 0, t^1_{n-1} \rangle & & x(2n-1) & \end{array}.$$

We now describe the moves in G^γ_0 following $\langle \hat{t}^1_{k(1)}, t^1_{k(1)} \rangle$ if $\hat{t}^1_{k(1)} = 1$. A less formal description is provided in the next paragraph. If II plays $\hat{t}^1_{k(1)} = 1$ and $\vec{t^1} = \langle t^1_j | j < k(1) \rangle$, then the moves of G^γ_0 following $\langle \hat{t}^1_{k(1)}, t^1_{k(1)} \rangle = \langle 1, - \rangle$ form the play of the auxiliary game

$$G_1^{\gamma-1}(T^1_{k(1)};\vec{t^1},\bar{x}(2k(1))),$$

described in Definition 2.3, (with integer moves $x(2k(1)+i)$ for $i \in \omega$). The moves of $G_1^{\gamma-1}(T^1_{k(1)};\vec{t^1},\bar{x}(2k(1)))$ are subject to similar conditions as those of $G_1^{\gamma-1}$. However, each t^j_n (defined below) for $1 < j \le \beta$ and each $\bar{x}(i)$ must belong to $T^1_{k(1)}$ and must be consistent with $\vec{t^1}$. Furthermore, ordinal auxiliary moves must be properly ordered in both games, but these conditions (on the ordinal auxiliary moves) are slightly different for $G_1^{\gamma-1}(T^1_{k(1)};\vec{t^1},\bar{x}(2k(1)))$ and $G_1^{\gamma-1}$. We review the play of G_0^γ for this case i.e. for $\hat{t}^1_{k(1)} = 1$.

Assume $\hat{t}^1_{k(1)} = 1$. Then for $j > k(1)$, no Borel auxiliary moves $(T^1_j;\langle \hat{t}^1_j, t^1_j\rangle)$ are played. Following the move $\langle \hat{t}^1_{k(1)}, t^1_{k(1)}\rangle = \langle 1, -\rangle$ are the integer moves $x(2k(1)+i)$ for $i \in \omega$ and a sequence $\langle (T^2_n;\langle \hat{t}^2_n, t^2_n\rangle) | \forall j < n \ \hat{t}^2_j = 0\rangle$ of Borel auxiliary moves which is related to B_2 via R_2. Player I may only play $T^2_n \in L(T^1_{k(1)})[\#^1_{\gamma-1}]$. Also, if $\forall j < i \ \hat{t}^2_j = 0$, then ordinal auxiliary moves λ^1_{2i} and λ^1_{2i+1} are respectively played with the integer moves $x(2k(1)+2i)$ and $x(2k(1)+2i+1)$ so that the λ^1_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega\cdot(k(1)+1)\rangle$ using $\langle \omega_{i+1}^{L(\#^1_\gamma(T^1_{k(1)}))} | i \le k(1)\rangle$. If II plays $\hat{t}^2_j = 0$ for all $j \in \omega$, then the play of G_0^γ is

$$\begin{array}{llllllll} \text{I} & T^1_0 & & x(0) & T^1_1 & & x(2) & \\ \text{II} & & \langle 0,t^1_0\rangle & x(1) & & \langle 0,t^1_1\rangle & x(3) & \cdots \end{array}$$

$$\cdots \begin{array}{llllll} T^1_{k(1)-1} & & x(2k(1)-2) & & T^1_{k(1)} & \\ & \langle 0,t^1_{k(1)-1}\rangle & & x(2k(1)-1) & & \langle 1,-\rangle \end{array}$$

$$\begin{array}{llllllll} T^2_0 & & x(2k(1)),\lambda^1_0 & & T^2_1 & & x(2k(1)+2),\lambda^1_2 & \\ & \langle 0,t^2_0\rangle & & x(2k(1)+1),\lambda^1_1 & & \langle 0,t^2_1\rangle & & x(2k(1)+3),\lambda^1_3 \ \cdots. \end{array}$$

If II plays $\hat{t}^2_{k(2)} = 1$ and $\vec{t^2} = \langle t^2_j | j < k(2)\rangle$, then the moves of G_0^γ following

$\langle \hat{t}^2_{k(2)}, t^2_{k(2)} \rangle = \langle 1, - \rangle$ form the play of the auxiliary game

$$G_1^{\gamma-2}(T^1_{k(1)}, T^2_{k(2)}; \vec{t^1}, \vec{t^2}, \bar{x}(2k(1)+2k(2)))$$

(with integer moves $x(2k(1)+2k(2)+i)$ for $i \in \omega$). Following the move $\langle \hat{t}^2_k, t^2_k \rangle = \langle 1, - \rangle$ is a sequence $\langle (T^3_n; \langle \hat{t}^3_n, t^3_n \rangle) | \forall j < n \ \hat{t}^3_j = 0 \rangle$ of Borel auxiliary moves which is related to B_3 via R_3. Player I may only play $T^3_n \in L(T^1_{k(1)}, T^2_{k(2)})[\#^1_{\gamma-2}]$. If $\forall j < i \ \hat{t}^3_j = 0$, then ordinal auxiliary moves λ^2_{2i} and λ^2_{2i+1} are respectively played with the integer moves $x(2k(1)+2k(2)+2i)$ and $x(2k(1)+2k(2)+2i+1)$ so that the λ^2_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (k(2)+1) \rangle$ using $\langle \omega_{i+1}^{L(\#^1_{\gamma-1}(T^1_{k(1)}, T^2_{k(2)}))} | i \leq k(2) \rangle$. If II plays $\hat{t}^3_j = 0$ for all $j \in \omega$ (and $\hat{t}^1_{k(1)} = \hat{t}^2_{k(2)} = 1$), then the play of G_0^γ is

$$\begin{array}{lcccccccc}
\text{I} & T^1_0 & & x(0) & & T^1_1 & & x(2) & \\
\text{II} & & \langle 0, t^1_0 \rangle & & x(1) & & \langle 0, t^1_1 \rangle & & x(3) \quad \cdots
\end{array}$$

$$\begin{array}{cccccc}
\cdots \; T^1_{k(1)-1} & & x(2k(1)-2) & & T^1_{k(1)} & \\
& \langle 0, t^1_{k(1)-1} \rangle & & x(2k(1)-1) & & \langle 1, - \rangle
\end{array}$$

$$\begin{array}{ccccccccc}
T^2_0 & & x(2k(1)), \lambda^1_0 & & T^2_1 & & x(2k(1)+2), \lambda^1_2 & & \\
& \langle 0, t^2_0 \rangle & & x(2k(1)+1), \lambda^1_1 & & \langle 0, t^2_1 \rangle & & x(2k(1)+3), \lambda^1_3 & \cdots
\end{array}$$

$$\begin{array}{cccccc}
\cdots \; T^2_{k(2)-1} & & x(2k(1)+2k(2)-2), \lambda^1_{2k(2)-2} & & T^2_{k(2)} & \\
& \langle 0, t^2_{k(2)-1} \rangle & & x(2k(1)+2k(2)-1), \lambda^1_{2k(2)-1} & & \langle 1, - \rangle
\end{array}$$

$$\begin{array}{ccccccccc}
T^3_0 & & x(2k(1)+2k(2)), \lambda^2_0 & & T^3_1 & & x(2k(1)+2k(2)+2), \lambda^2_2 & & \\
& \langle 0, t^3_0 \rangle & & x(2k(1)+2k(2)+1), \lambda^2_1 & & \langle 0, t^3_1 \rangle & & x(2k(1)+2k(2)+3), \lambda^2_3 & \cdots .
\end{array}$$

If II plays $\hat{t}^3_{k(3)} = 1$, then we continue as above.

In general, for every play of G_0^γ, either

(1) there exists δ such that $0 \leq \delta < \gamma$ and for all $k \in \omega$, Borel auxiliary moves $\langle \hat{t}^{\delta+1}_k, t^{\delta+1}_k \rangle = \langle 0, t^{\delta+1}_k \rangle$ are played, or

(2) for each δ such that $1 \leq \delta \leq \gamma$, a Borel auxiliary move $\langle \hat{t}^\delta_{k(\delta)}, t^\delta_{k(\delta)} \rangle = \langle 1, - \rangle$ is played.

Now we give a general description of the play of G_0^γ, in which we (eventually) refer to the cases (1) and (2) just described. We have described the play of G_0^γ for case (1) when $\delta \in \{0,1,2\}$.

Suppose $\delta < \gamma$ and we have reached a legal position

$$p = (T_0^1; \langle 0, t_0^1\rangle; x(0); x(1); \ldots\ldots$$
$$\ldots\ldots; T_{k(1)}^1; \langle 1, -\rangle; T_0^2; \langle 0, t_0^2\rangle; x(2k(1)), \lambda_0^1; x(2k(1)+1), \lambda_1^1;$$
$$T_1^2; \langle 0, t_1^2\rangle; x(2k(1)+2), \lambda_2^1; x(2k(1)+3), \lambda_3^1; \ldots\ldots ; T_{k(2)}^2; \langle 1, -\rangle; \ldots$$
$$\ldots\ldots$$
$$\ldots\ldots; T_0^\delta; \langle 0, t_0^\delta\rangle; x(2k(\delta-1)), \lambda_0^{\delta-1}; x(2k(\delta-1)+1), \lambda_1^{\delta-1};$$
$$T_1^\delta; \langle 0, t_1^\delta\rangle; x(2k(\delta-1)+2), \lambda_2^{\delta-1}; x(2k(\delta-1)+3), \lambda_3^{\delta-1}; \ldots\ldots ; T_{k(\delta)}^\delta; \langle 1, -\rangle)$$

of G_0^γ. Then the moves of G_0^γ following $\langle \hat{t}_{k(\delta)}^\delta, t_{k(\delta)}^\delta\rangle = \langle 1, -\rangle$ form the play of the auxiliary game

$$G_1^{\gamma-\delta}(\langle T_{k(j)}^j | j \leq \delta\rangle; \langle t_k^j | j \leq \delta \ \&\ k < k(j)\rangle, \sum_{i=1}^{\delta} \bar{x}(2k(i)))$$

(with integer moves $x(2k(1)+\cdots+2k(\delta)+i)$ for $i \in \omega$). Following the move $\langle \hat{t}_{k(\delta)}^\delta, t_{k(\delta)}^\delta\rangle = \langle 1, -\rangle$ is a sequence

$$\langle (T_n^{\delta+1}; \langle \hat{t}_n^{\delta+1}, t_n^{\delta+1}\rangle) | \forall j < n\ \hat{t}_j^{\delta+1} = 0\rangle$$

of Borel auxiliary moves which is related to $B_{\delta+1}$ via $R_{\delta+1}$. Let

$$\vec{T}_j = \langle T_k^i | \hat{t}_k^i = 1 \ \&\ i \leq j\rangle,\ \vec{t^j} = \langle t_k^j | \hat{t}_k^j = 0\rangle, \text{ and } \vec{t}_j = \langle t_k^i | i \leq j \ \&\ \hat{t}_k^i = 0\rangle,$$

(for any j). Then $\vec{T}_\delta = \langle T_{k(j)}^j | 1 \leq j \leq \delta\rangle$. Player I may only play $T_n^{\delta+1} \in L(\vec{T}_\delta)[\#_{\gamma-\delta}^1]$. If $\forall j < i\ \hat{t}_j^{\delta+1} = 0$, then ordinal auxiliary moves λ_{2i}^δ and λ_{2i+1}^δ are respectively played with the integer moves $x(2k(1)+\cdots+2k(\delta)+2i)$ and

$x(2k(1)+\cdots+2k(\delta)+2i+1)$ so that the λ_i^δ's are properly ordered with respect to $\langle A_\alpha|\alpha<\omega\cdot(k(\delta)+1)\rangle$ using $\langle\omega_{i+1}^{L(\#^1_{\gamma-\delta+1}(T^1_{k(1)},T^2_{k(2)},\ldots,T^\delta_{k(\delta)}))}|i\le k(\delta)\rangle$.

Case 1. If II plays $\hat{t}_j^{\delta+1}=0$ for all $j\in\omega$, then the play of G_0^γ is

$$\begin{array}{llllllll} \text{I} & T_0^1 & & x(0) & & T_1^1 & & x(2) \quad \ldots \\ \text{II} & & \langle 0,t_0^1\rangle & & x(1) & & \langle 0,t_1^1\rangle & \quad x(3) \end{array}$$

$$\begin{array}{llllll} \ldots & T^1_{k(1)-1} & & x(2k(1)-2) & & T^1_{k(1)} \\ & & \langle 0,t^1_{k(1)-1}\rangle & & x(2k(1)-1) & \quad \langle 1,-\rangle \end{array}$$

$$\begin{array}{llllllll} T_0^2 & & x(2k(1)),\lambda_0^1 & & T_1^2 & & x(2k(1)+2),\lambda_2^1 & \ldots \\ & \langle 0,t_0^2\rangle & & x(2k(1)+1),\lambda_1^1 & & \langle 0,t_1^2\rangle & & x(2k(1)+3),\lambda_3^1 \end{array}$$

....

$$\begin{array}{llllll} \ldots & T^\delta_{k(\delta)-1} & & x(2k(1)\cdots+2k(\delta)-2),\lambda^{\delta-1}_{2k(\delta)-2} & & T^\delta_{k(\delta)} \\ & & \langle 0,t^\delta_{k(\delta)-1}\rangle & & x(2k(1)+\cdots+2k(\delta)-1),\lambda^{\delta-1}_{2k(\delta)-1} & \quad \langle 1,-\rangle \end{array}$$

$$\begin{array}{llllllll} T_0^{\delta+1} & & x(2k(1)+\cdots+2k(\delta)),\lambda_0^\delta & & T_1^{\delta+1} & & x(2k(1)+\cdots+2k(\delta)+2),\lambda_2^\delta & \ldots \\ & \langle 0,t_0^{\delta+1}\rangle & & x(2k(1)+\cdots+2k(\delta)+1),\lambda_1^\delta & & \langle 0,t_1^{\delta+1}\rangle & & x(2k(1)+\cdots+2k(\delta)+3),\lambda_3^\delta \end{array}$$

Case 2. Suppose we reach a position p whose last move is $\langle\hat{t}^\gamma_{k(\gamma)},t^\gamma_{k(\gamma)}\rangle=\langle 1,-\rangle$. Then for $1\le\delta\le\gamma$, p contains a Borel auxiliary move $\langle\hat{t}^\delta_{k(\delta)},t^\delta_{k(\delta)}\rangle=\langle 1,-\rangle$. p has the following form:

$$p=(T_0^1;\langle 0,t_0^1\rangle;x(0);x(1);...;T^1_{k(1)};\langle 1,-\rangle;T_0^2;\langle 0,t_0^2\rangle;x(2k(1)),\lambda_0^1;x(2k(1)+1),\lambda_1^1;$$
$$T_1^2;\langle 0,t_1^2\rangle;x(2k(1)+2),\lambda_2^1;x(2k(1)+3),\lambda_3^1;...;T^2_{k(2)};\langle 1,-\rangle;\ldots\ldots$$
$$\ldots\ldots$$
$$\ldots\ldots;T_0^\delta;\langle 0,t_0^\delta\rangle;x(2k(\delta-1)),\lambda_0^{\delta-1};x(2k(\delta-1)+1),\lambda_1^{\delta-1};$$
$$T_1^\delta;\langle 0,t_1^\delta\rangle;x(2k(\delta-1)+2),\lambda_2^{\delta-1};x(2k(\delta-1)+3),\lambda_3^{\delta-1};\ldots\ldots;T^\delta_{k(\delta)};\langle 1,-\rangle)$$
$$\ldots\ldots$$
$$\ldots\ldots;T_0^\gamma;\langle 0,t_0^\gamma\rangle;x(2k(\gamma-1)),\lambda_0^{\gamma-1};x(2k(\gamma-1)+1),\lambda_1^{\gamma-1};$$
$$T_1^\gamma;\langle 0,t_1^\gamma\rangle;x(2k(\gamma-1)+2),\lambda_2^{\gamma-1};x(2k(\gamma-1)+3),\lambda_3^{\gamma-1};\ldots;T^\gamma_{k(\gamma)};\langle 1,-\rangle).$$

Let

$$\vec{T} = \vec{T}_\gamma = \langle T^\delta_j | \hat{t}^\delta_j = 1 \ \&\ 1 \le \delta \le \gamma\rangle \text{ and } \vec{t} = \vec{t}_\gamma = \langle t^\delta_j | \hat{t}^\delta_j = 0 \ \&\ 1 \le \delta \le \gamma\rangle.$$

Then $\vec{T} = \langle T^\delta_{k(\delta)} | 1 \le \delta \le \gamma\rangle$ and $\vec{t} = \langle t^\delta_k | 1 \le \delta \le \gamma \ \&\ k < k(\delta)\rangle$. Each move of G^γ_0 following $\langle \hat{t}^\gamma_{k(\gamma)}, t^\gamma_{k(\gamma)}\rangle = \langle 1, -\rangle$ consists of an integer $x(2k(1) + \cdots + 2k(\delta) + i)$ and an ordinal ξ_i. The ξ_i's must be properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (k(\gamma)+1)\rangle$ using $\langle \omega^{L(\#^1_1(\vec{T}))}_{i+1} | i \le k(\gamma)\rangle$. Whenever $\hat{t}^\delta_{k(\delta)} = 1$ for $1 \le \delta \le \gamma$, the play of G^γ_0 is

$$\begin{array}{lllllll}
\text{I} & T^1_0 & & x(0) & & T^1_1 & & x(2) & & \cdots \\
\text{II} & & \langle 0, t^1_0\rangle & & x(1) & & \langle 0, t^1_1\rangle & & x(3) &
\end{array}$$

$$\begin{array}{lllllll}
\cdots & T^1_{k(1)-1} & & x(2k(1)-2) & & T^1_{k(1)} & \\
& & \langle 0, t^1_{k(1)-1}\rangle & & x(2k(1)-1) & & \langle 1, -\rangle
\end{array}$$

$$\begin{array}{lllllll}
T^2_0 & & x(2k(1)), \lambda^1_0 & & T^2_1 & & x(2k(1)+2), \lambda^1_2 & & \cdots \\
& \langle 0, t^2_0\rangle & & x(2k(1)+1), \lambda^1_1 & & \langle 0, t^2_1\rangle & & x(2k(1)+3), \lambda^1_3 &
\end{array}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{array}{lllllll}
\cdots & T^\delta_{k(\delta)-1} & & x(2k(1) \cdots + 2k(\delta)-2), \lambda^{\delta-1}_{2k(\delta)-2} & & T^\delta_{k(\delta)} & \\
& & \langle 0, t^\delta_{k(\delta)-1}\rangle & & x(2k(1) + \cdots + 2k(\delta)-1), \lambda^{\delta-1}_{2k(\delta)-1} & & \langle 1, -\rangle
\end{array}$$

$$\begin{array}{lllllll}
T^{\delta+1}_0 & & x(2k(1) + \cdots + 2k(\delta)), \lambda^\delta_0 & & T^{\delta+1}_1 & & x(2k(1) + \cdots + 2k(\delta)+2), \lambda^\delta_2 & & \cdots \\
& \langle 0, t^{\delta+1}_0\rangle & & x(2k(1) + \cdots + 2k(\delta)+1), \lambda^\delta_1 & & \langle 0, t^{\delta+1}_1\rangle & & x(2k(1) + \cdots + 2k(\delta)+3), \lambda^\delta_3 &
\end{array}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{array}{lllllll}
\cdots & T^\gamma_{k(\gamma)-1} & & x(2k(1) + \cdots + 2k(\gamma)-2), \lambda^{\gamma-1}_{2k(\gamma)-2} & & T^\gamma_{k(\gamma)} & \\
& & \langle 0, t^\gamma_{k(\gamma)-1}\rangle & & x(2k(1) + \cdots + 2k(\gamma)-1), \lambda^{\gamma-1}_{2k(\gamma)-1} & & \langle 1, -\rangle
\end{array}$$

$$\begin{array}{lllll}
x(2k(1) + \cdots + 2k(\gamma)), \xi_0 & & x(2k(1) + \cdots + 2k(\gamma)+2), \xi_2 & & \cdots. \\
& x(2k(1) + \cdots + 2k(\gamma)+1), \xi_1 & & x(2k(1) + \cdots + 2k(\gamma)+3), \xi_3 &
\end{array}$$

Recall that conditions on λ^δ_i and T^δ_i for $1 \le \delta \le \gamma$ are described just before Case 1.

Player I wins G^γ_0 iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. G^γ_0 is an open game and therefore we define, for each ordinal α, P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in \mathrm{ON}} P_\alpha$. If p is a legal position in G^γ_0, let ℓ_p denote the set of legal positions in G_0^γ consistent with p. The set ℓ of legal positions for G_0^γ is in $L[\#_\gamma^1]$.

Use $\langle P_\alpha | \alpha \in ON\rangle$ and Theorem 0.14 to define a wellordering $\prec$ of ℓ and the canonical w.s. s_0^γ for G_0^γ so that the following hold: s_0^γ is in $L[\#_\gamma^1]$, and if p is a legal position in G_0^γ, then $s_0^\gamma|\ell_p$ is a w.s. for $(G_0^\gamma)_p$ and is definable in any inner model of ZF in which $\prec|\ell_p$ is definable. By induction, s_0^γ has the following properties:

Lemma 2.2.1. Let p be a legal position in G_0^γ. Recall $\#_0^1(U) = U$ for any U. Then $s_0^\gamma|\ell_p$ is a w.s. for $(G_0^\gamma)_p$ and has the following property:

iii.) If p includes the move $\langle \hat{t}^j_{k(j)}, t^j_{k(j)}\rangle = \langle 1, -\rangle$ for $1 \le j \le \delta$, then

$$\vec{T}_\delta = \langle T^j_{k(j)} | 1 \le j \le \delta\rangle,$$

and $s_0^\gamma|\ell_p$ is definable in $L(\vec{T}_\delta)[\#^1_{\gamma-\delta}]$ from $\langle \omega_{i+1}^{L(\#^1_{\gamma-\delta+1}(\vec{T}_\delta))} | i \le k(\delta)\rangle$.

In particular,

iv.) if p includes the move $\langle \hat{t}^1_{k(1)}, t^1_{k(1)}\rangle = \langle 1, -\rangle$, then $s_0^\gamma|\ell_p$ is definable in $L(T^1_{k(1)})[\#^1_{\gamma-1}]$ from $\langle \omega_{i+1}^{L(\#^1_\gamma(T^1_{k(1)}))} | i \le k(1)\rangle$.

By Lemma 2.2.1 and the Induction Hypothesis, we have the following:

Lemma 2.2.2. Let $k_1 = k(1) \in \omega$ and let B be the set witnessed to be $((\gamma-1) * \Pi^0_1)^*_+$ by $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_2$, $\langle A_\alpha | \alpha < \omega^2\rangle$, and k_1. Suppose

$p = (T_0^1; \langle 0, t_0^1\rangle; x(0); x(1); T_1^1; \langle 0, t_1^1\rangle; x(2); x(3); ...$

$...;\mathrm{T}^1_{k(1)-1}; \langle 0, t^1_{k(1)-1}\rangle; x(2k_1-2); x(2k_1-1); T^1_{k(1)}; \langle 1, -\rangle)$

is a legal position of G_β^γ. Then $\vec{t^1} = \langle t_i^1 | 0 \le i < k(1)\rangle$. If $B_1(x, k_1)$, then

$$x \in B(T^1_{k(1)}; \vec{t^1}, \bar{x}(2k_1)) \Leftrightarrow x \in A(T^1_{k(1)}; \vec{t^1}, \bar{x}(2k_1)).$$

Moreover, $s_0^\gamma|\ell_p$ can be integrated so as to obtain a w.s. $s' \in L(\#_\gamma^1(T_{k(1)}^1))$ for $A(T_{k(1)}^1; \vec{t}^1, \bar{x}(2k_1))$ such that

v.) s' is a w.s. of the player for whom s_0^γ is a w.s.

vi.) If s' is a w.s. for I and x is a play consistent with s', then x is a play consistent with $T_{k(1)}^1$ so that $x \in B(\vec{t}^1, \bar{x}(2k_1))$. If in addition $B_1(x, k_1)$, then $x \in A(\vec{t}^1, \bar{x}(2k_1))$.

vii.) If p is a position of the game $A(T_{k(1)}^1; \vec{t}^1, \bar{x}(2k_1))$ such that the moves in p of the player for whom s' is not a w.s. are consistent with $\vec{t}^1$ and $\bar{x}(2k_1)$, then p is consistent with $\vec{t}^1$ and $\bar{x}(2k_1)$.

Claim: G_A has a w.s. $s \in L[\#_\gamma^1]$ and s is a w.s. of the player for whom s_0^γ is a w.s.

To obtain the proof of this claim, make the following changes to the proofs of Claims I and II of Theorem 2.0: Replace each U_i by T_i^1, U_i' by $\tilde{T}_i^1$, $\hat{u}_i$ by $\hat{t}_i^1$, u_i by t_i^1, s_0^2 by s_0^γ, G_0^2 by G_0^γ, $L[\#_2^1]$ by $L[\#_\gamma^1]$, and $L(\#_1^1(U_i))$ by $L(\#_{\gamma-1}^1(T_i^1))$. We now give the details of the proof of the Claim:

Claim I: Player I has a w.s. for G_A if he has one for G_0^γ.

Let's first consider the case in which $\langle\ \rangle \in P$. Then $s_0^\gamma \in L[\#_\gamma^1]$ is a w.s. for I in G_0^γ. We use s_0^γ to define a w.s. s for I in G_A. Let

$$T_0^1 = s_0^\gamma(\langle\ \rangle),\ \langle \hat{t}_0^1, t_0^1 \rangle = \langle 1, - \rangle, \text{ and } p_0 = (T_0^1; \langle 1, - \rangle).$$

By Lemma 2.2.2(v), obtain a w.s. $s_0 \in L(\#_{\gamma-1}^1(T_0^1))$ for $A(T_0^1)$ by integrating the w.s. $s_0^\gamma|\ell_{p_0}$ for $(G_0^\gamma)_{p_0}$, and let $s(p) = s_0(p)$ for any position

$p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 0)$. If $R_1(\bar{x}(i), 0)$ holds for all i, then $x \in A(T_0^1)$ so that $x \in A$ by Lemma 2.2.2(vi).

Suppose we reach a position such that $\neg R_1(\bar{x}(i_0), 0)$ and $\forall j < i_0 R_1(\bar{x}(j), 0)$. We have defined

$$s(x(1); x(3); ...; x(2j-1)) = x(2j) \text{ for } 2j < i_0. \qquad \text{(viii)}$$

Since i_0 is odd by (ii), let $\langle \hat{t}_0^1, t_0^1 \rangle = \langle 0, \bar{x}(i_0) \rangle$ and $p_0' = (T_0^1; \langle 0, t_0^1 \rangle)$. Define $x(0) = s(\ )$ to be $s_0^\gamma(p_0')$ and T_1^1 to be $s_0^\gamma(p_0' * (x(0); x(1)))$. Let

$$p_1 = p_0' * (x(0); x(1); T_1^1; \langle 1, - \rangle).$$

By Lemma 2.2.2(v), obtain a w.s. $s_1 \in L(\#_{\gamma-1}^1(T_1^1))$ for $A(T_1^1; t_0^1, \bar{x}(2))$ by integrating the w.s. $s_0^\gamma | \ell_{p_1}$ for $(G_0^\gamma)_{p_1}$, and let $s(p) = s_1(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 1)$. By Lemma 2.2.2(vii), this definition of s is consistent with (viii). If $R_1(\bar{x}(i), 1)$ holds at every position, then $x \in A(T_1^1; t_0^1, \bar{x}(2))$ so that $x \in A$ by Lemma 2.2.2. If we reach a position such that $\neg R_1(\bar{x}(i_1), 1)$ and $\forall j < i_1 R_1(\bar{x}(j), 1)$, then continue to define s by integrating s_0^γ in the same manner as above.

In general, suppose we reach a position such that

$$\neg R_1(\bar{x}(i_j), j) \text{ and } \forall i < i_j R_1(\bar{x}(i), j)$$

for $j = 1, 2, 3, ..., k-1$. Then let $x(0), x(2), x(4), ..., x(2k)$ and $T_0^1, T_1^1, T_2^1, ..., T_k^1$ be such that the position

$$p_k = (T_0^1; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); T_1^1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); ...$$
$$...; T_{k-1}^1; \langle 0, \bar{x}(i_{k-1}) \rangle; x(2k-2); x(2k-1); T_k^1; \langle 1, - \rangle)$$

is consistent with s_0^γ. By Lemma 2.2.2(v), obtain a w.s. $s_k \in L(\#_\gamma^1(T_k^1))$ for

$$A(T_k^1; t_0^1, t_1^1, ...t_{k-1}^1, \bar{x}(2k))$$

by integrating the w.s. $s_0^\gamma | \ell_{p_k}$ for $(G_0^\gamma)_{p_k}$, and let

$$s(p) = s_k(p)$$

for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), k)$.

Claim: The strategy s of player I is a w.s. for G_A.

Let x be a play of G_A consistent with s. First consider the case in which there is a least k such that $B_1(x, k)$. Then for each $j < k$, there is a least i_j such that $\neg R_1(\bar{x}(i_j), j)$. By the definition of s, there exist $T_0^1, T_1^1, T_2^1, ..., T_k^1$ such that the position

$$p_k = (T_0^1; \langle 0, \bar{x}(i_0)\rangle; x(0); x(1); T_1^1; \langle 0, \bar{x}(i_1)\rangle; x(2); x(3); ...$$
$$...; T_{k-1}^1; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2); x(2k-1); T_k^1; \langle 1, -\rangle)$$

is consistent with s_0^γ. We obtained the w.s. s_k for

$$A(T_k^1; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k))$$

by integrating the w.s. $s_0^\gamma | \ell_{p_k}$ for $(G_0^\gamma)_{p_k}$ and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), k)$. Since $B_1(x, k)$, $\forall j R_1(\bar{x}(j), k)$. Therefore, x is a play consistent with s_k so that

$$x \in A(T_k^1; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k)).$$

Clearly, x is consistent with each $\bar{x}(i_j)$ and by Lemma 2.2.2(vi), $\forall j\ \bar{x}(j) \in T_k^1$. Thus, $x \in A$ if $\exists k B_1(x, k)$.

Now assume $\neg B_1(x, k)$ for all k. Then for each k, there is a least i_k such that $\neg R_1(\bar{x}(i_k), k)$. Moreover, for each k, we obtained the position

$$p_k = (T^1_0; \langle 0, \bar{x}(i_0)\rangle; x(0); x(1); T^1_1; \langle 0, \bar{x}(i_1)\rangle; x(2); x(3); ...$$

$$...; T^1_{k-1}; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2); x(2k-1); T^1_k; \langle 1, -\rangle)$$

consistent with s^γ_0. Since each p_k is consistent with s^γ_0, the play

$$y = (T^1_0; \langle 0, \bar{x}(i_0)\rangle; x(0); x(1); T^1_1; \langle 0, \bar{x}(i_1)\rangle; x(2); x(3); T^1_2; \langle 0, \bar{x}(i_2)\rangle; x(4); x(5); ...$$

$$...; T^1_n; \langle 0, \bar{x}(i_n)\rangle; x(2n); x(2n+1); ...)$$

is consistent with s^γ_0. Since y is a play of G^γ_0 in which player II made only legal moves, y is a win for II, contradicting s^γ_0 being a w.s. for I. Thus, $\exists k B_1(x, k)$, $x \in A$, and s is a w.s. for player I.

Claim II: Player II has a w.s. for G_A if he has one for G^γ_0.

Now let's consider the case $\langle\ \rangle \notin$ P. We integrate II's w.s. $s^\gamma_0 \in L[\#^1_\gamma]$ for G^γ_0 to get the w.s. s for II in G_A. Let

$$T^1_0 = \{\text{positions } t \text{ in } G_A | \forall T' \in L[\#^1_\gamma]\ \langle 0, t\rangle \neq s^\gamma_0(T')\}.$$

Then $\forall t \in T^1_0 \forall T' \in L[\#^1_\gamma]\ \langle 0, t\rangle \neq s^\gamma_0(T')$. (ix)

x.) If $(T'; \langle 0, t\rangle)$ is a legal position of G^γ_0, then for each ordinal α, $P_\alpha \cap \ell_{(T';\langle 0,t\rangle)}$ is definable in $L[\#^1_\gamma]$.

Therefore, $T^1_0 \in L[\#^1_\gamma]$. Also, by (ix), $\langle 1, -\rangle = s^\gamma_0(T^1_0)$. Let $p_0 = (T^1_0, \langle 1, -\rangle)$. By Lemma 2.2.2(v), obtain a w.s. $s_0 \in L(\#^1_{\gamma-1}(T^1_0))$ for $A(T^1_0)$ by integrating the w.s. $s^\gamma_0 | \ell_{p_0}$ for $(G^\gamma_0)_{p_0}$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i\ \bar{x}(j) \in T^1_0$. If $\bar{x}(i) \in T^1_0$ holds at every position, then $x \notin A(T^1_0)$ so that $x \notin A$.

Suppose we reach a position such that $\bar{x}(i_0) \notin T_0^1$ and $\forall j < i_0\ \bar{x}(j) \in T_0^1$. We have defined

$$s(x(0); x(2); ...; x(2j)) = x(2j+1) \text{ for } 2j < i_0. \tag{xi}$$

Since $\bar{x}(i_0) \notin T_0^1$, there exists $\tilde{T}_0^1 \in L[\#_\gamma^1]$ such that $\langle 0, \bar{x}(i_0)\rangle = s_0^\gamma(\tilde{T}_0^1)$.

Let $\langle \hat{t}_0^1, t_0^1\rangle = \langle 0, \bar{x}(i_0)\rangle$ and $p_0' = (\tilde{T}_0^1; \langle 0, t_0^1\rangle; x(0))$. Define $x(1) = s(x(0))$ to be $s_0^\gamma(p_0')$. Let

$$T_1^1 = \{\text{positions } t \text{ in } G_A(\bar{x}(2), t_0^1) | \forall T' \in L[\#_\gamma^1]\ \langle 0, t\rangle \neq s_0^\gamma(\tilde{T}_0^1; x(0); T')\}.$$

Then $T_1^1 \in L[\#_\gamma^1]$ and $\langle 1, -\rangle = s_0^\gamma(\tilde{T}_0^1; x(0); T_1^1)$. Let

$$p_1 = p_0' * (x(1); T_1^1; \langle 1, -\rangle).$$

By Lemma 2.2.2(v), obtain a w.s. s_1 for $A(T_1^1; t_0^1, \bar{x}(2))$ by integrating the w.s. $s_0^\gamma | \ell_{p_1}$ for $(G_0^\gamma)_{p_1}$, and let $s(p) = s_1(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 1)$. By Lemma 2.2.2(vi), this definition of s is consistent with (xi). If $\bar{x}(i) \in T_1^1$ holds at every position, then $x \notin A(T_1^1; t_0^1, \bar{x}(2))$ so that $x \notin A$. If we reach a position such that $\bar{x}(i_1) \notin T_1^1$, then continue to define s by integrating s_0^γ in the same manner as above.

In general, suppose we reach a position such that

$$\bar{x}(i_j) \notin T_j^1 \text{ and } \forall i < i_j\ \bar{x}(i) \in T_j^1 \text{ for } j = 1, 2, 3, ..., k-1.$$

Then let $x(0), x(2), x(4), ..., x(2k-2)$ and $\tilde{T}_0^1, \tilde{T}_1^1, \tilde{T}_2^1, ..., \tilde{T}_{k-1}^1$ be such that the position

$$p_{k-1}' = (\tilde{T}_0^1; \langle 0, \bar{x}(i_0)\rangle; x(0); x(1); \tilde{T}_1^1; \langle 0, \bar{x}(i_1)\rangle; x(2); x(3); ...$$
$$...; \tilde{T}_{k-1}^1; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2))$$

is consistent with s_0^γ. Let

$$T_k^1 = \{\text{positions } t \text{ in } G_A \text{ consistent with } \bar{x}(2k) \text{ and } \langle \bar{x}(i_j) | j < k \rangle |$$
$$\forall T' \in L[\#_\gamma^1]\ \langle 0, t \rangle \neq s_0^\gamma(p'_{k-1} * (x(2k-1); T'))\}.$$

Then $p_k = p'_{k-1} * (x(2k-1); T_k^1; \langle 1, - \rangle)$ is consistent with s_0^γ. By Lemma 2.2.2(v), obtain a w.s. s_k for $A(T_k^1; t_0^1, t_1^1, ..., t_{k-1}^1, \bar{x}(2k))$ by integrating the w.s. $s_0^\gamma | \ell_{p_k} \in L(\#_\gamma^1(T_k^1))$ for $(G_0^\gamma)_{p_k}$, and let

$$s(p) = s_k(p)$$

for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i\ \bar{x}(j) \in T_k^1$.

Claim: The strategy s of player II is a w.s. for G_A.

Let x be a play of G_A consistent with s. First consider the case in which there is a least k such that $\forall i\ \bar{x}(i) \in T_k^1$. Then for each $j < k$, there is a least i_j such that $\bar{x}(i_j) \notin T_j^1$. By the definition of $x(2j+1) = s(x(0); x(2); ...; x(2j))$ for $j < k$ and by the definition of the T_i^1's, there exist $\tilde{T}_0^1, \tilde{T}_1^1, \tilde{T}_2^1, ..., \tilde{T}_{k-1}^1$ such that the position

$$p_k = (\tilde{T}_0^1; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); \tilde{T}_1^1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); ...$$
$$...; \tilde{T}_{k-1}^1; \langle 0, \bar{x}(i_{k-1}) \rangle; x(2k-2); x(2k-1); T_k^1; \langle 1, - \rangle)$$

is consistent with s_0^γ. We obtained the w.s. s_k for

$$A(T_k^1; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k))$$

by integrating the w.s. $s_0^\gamma | \ell_{p_k}$ for $(G_0^\gamma)_{p_k}$ and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i\ \bar{x}(j) \in T_k^1$. Since we are assuming $\forall j\ \bar{x}(j) \in T_k^1$, x is a play consistent with s_k so that

$$x \notin A(T^1_k; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k)).$$

Therefore, $x \notin A$.

Now assume for each k, there is a least i_k such that $\bar{x}(i_k) \notin T^1_k$. By the definition of $x(2j+1) = s(x(0); x(2); ...; x(2j))$ for $j < k$ and by the definition of the T^1_i's, there exist $\tilde{T}^1_0, \tilde{T}^1_1, \tilde{T}^1_2, ..., \tilde{T}^1_{k-1}$ such that the position

$$(\tilde{T}^1_0; \langle 0, \bar{x}(i_0)\rangle; x(0); x(1); \tilde{T}^1_1; \langle 0, \bar{x}(i_1)\rangle; x(2); x(3); ...; \tilde{T}^1_k; \langle 0, \bar{x}(i_k)\rangle)$$

is consistent with s^γ_0. Since each such position is consistent with s^γ_0, $\neg R_1(\bar{x}(i_k), k)$. Therefore, $\forall k \neg B(x, k)$ so that x is a win for II.

Consequently, s is a w.s. in G_A of the player for whom s^γ_0 is a w.s. ∎

Now we generalize the auxiliary game G^γ_0:

Definition 2.2. Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi^0_1$, and $\langle A_\alpha \mid \alpha < \omega^2\rangle$ strongly witness $A \in (\gamma * \Pi^0_1)^*$. Then we refer to the auxiliary game G^γ_0 described in the Proof of Theorem 2.2 as

the G^γ_0 auxiliary game determined by $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi^0_1$, and $\langle A_\alpha \mid \alpha < \omega^2\rangle$.

Suppose $\vec{U} = \langle U_i | i < \beta\rangle$ and $\vec{u} = \langle u_i | i < \gamma\rangle$ respectively are a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of G_A. Then *the $G^\gamma_0(\vec{U}; \vec{u})$ auxiliary game determined by*

$$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1, \text{ and } \langle A_\alpha \mid \alpha < \omega^2\rangle$$

is the game in which player I wins iff a position is reached at which II cannot make a (legal) move, which has exactly the same moves as G^γ_0, and these moves are subject to the following conditions:

i.) The sequence $\langle (T_n^\delta; \langle \hat{t}_n^\delta, t_n^\delta \rangle) | \forall j < n \ \hat{t}_j^\delta = 0 \rangle$ of Borel auxiliary moves and the Π_1^0 set B_δ are related via R_{B_δ}.

ii.) Each $\bar{x}(i) \in \bigcap_{j<\beta} U_j$ and each $\bar{x}(i)$ must be consistent with every u_j.

iii.) $T_i^\delta \in L(\vec{U}, \vec{T}_{\delta-1})[\#_{\gamma+1-\delta}^1]$.

iv.) The λ_i^δ's are properly ordered with repect to $\langle A_\alpha | \alpha < \omega \cdot (k(\delta)+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_{\gamma+1-\delta}^1(\vec{U}, \vec{T}_\delta))} | i \leq k(\delta) \rangle$. $(\delta \geq 1.)$

v.) If $\hat{t}_{k(\gamma)}^\gamma = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (k(\gamma)+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{U}, \vec{T}))} | i \leq k(\gamma) \rangle$.

These conditions are analogous to the conditions for the moves of G_0^γ. The first is a condition which the moves of G_0^γ also must satisfy. The others are derived by changing the conditions for the moves of G_0^γ so that we obtain conditions which are consistent with $\vec{U}$ and $\vec{u}$. We refer to $G_0^\gamma(\vec{U}; \vec{u})$ instead of the $G_0^\gamma(\vec{U}; \vec{u})$ auxiliary game determined by $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1$, and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$ whenever $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1$ and $\langle A_\alpha \mid \alpha < \omega^2 \rangle$ are clear from the context. Analogous to Theorem 2.2, we have the following:

Corollary 2.2.1. Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, A, $\vec{U}$, and $\vec{u}$ be as in Definition 2.2. Let p be a legal position of a game G^* such that the moves of G^* following p constitute a play of $G_0^\gamma(\vec{U}, \vec{u})$. Suppose $\vec{U}$ has a wellordering which is definable in $L(\vec{U})$, $\mathrm{L}(\vec{U})[\#_\gamma^1] \models$ "$r^{\#_\gamma^1}$ exists for every real r," and s^* is a w.s. for G^* such that $s^* | \ell_p \in L(\vec{U})[\#_\gamma^1]$. Then $s^* | \ell_p$ can be integrated so as to obtain a w.s. $s_p \in L(\vec{U})[\#_\gamma^1]$ for $A(\vec{U}; \vec{u})$ such that the following hold:

i.) s_p is a w.s. of the player for whom s^* is a w.s.,

ii.) If s_p is a w.s. for I, $\hat{p}$ is a position consistent with s_p, and the moves in $\hat{p}$ of the player for whom s_p is not a w.s. are consistent with $\vec{u}$, then $\hat{p} \in \bigcap_{i<\beta} U_i$. Therefore, if s_p is a w.s. for I and x is a play consistent with s_p, then $x \in A(\vec{u})$.

iii.) Let $\hat{p}$ be a position consistent with s_p and with $\vec{U}$. If the moves in $\hat{p}$ of the player for whom s_p is not a w.s. are consistent with $\vec{u}$, then $\hat{p}$ is consistent with $\vec{u}$. ■