Now we prove a generalization of Theorem 2.1. In the following proof, we integrate a w.s. $s_1^\gamma$ with respect to some ordinal auxiliary moves $\lambda_i^0$ in a similar manner to the way in which the w.s. $s_1^2$ of Theorem 2.1 is integrated with respect to the moves $\lambda_i^0$ of $G_0^2$. Otherwise, $s_1^\gamma$ is integrated analogous to the manner in which $s_0^\gamma$ of Theorem 2.2 is integrated.

**Theorem 2.3.** If $L[\#_\gamma^1]$ has indiscernibles, then $\mathrm{Det}(\gamma * \Pi_1^0)_+^*$.

**Proof:** Assume $L[\#_\gamma^1]$ has an uncountable set $C_1^\gamma$ of indiscernibles. Let

$$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi_1^0,\ \langle A_\alpha \mid \alpha < \omega^2\rangle, \text{ and } D$$

strongly witness $A \in (\gamma * \Pi_1^0)_+^*$, and let $\langle D_\alpha | \alpha < \omega \cdot m\rangle$ witness that $D$ is $\omega \cdot m - \Pi_1^0$, where $m \in \mathbf{N}$. Then for $1 \leq i \leq \beta$, there exist $R_i \in \Delta_1^0$ such that

i.) $B_i(x, n) \leftrightarrow \forall k R_i(\bar{x}(k), n)$, and

ii.) if $\neg R_i(\bar{x}(k), n)$ and $\forall j < k R_i(\bar{x}(j), n)$, then $k$ is odd.

We show that $G_A$ has a w.s. $s$. Condition (ii) helps to simplify the proof.

We describe an open game $G_1^\gamma$ which has a w.s. $s_1^\gamma \in L[\#_\gamma^1]$. We integrate $s_1^\gamma$ to get the w.s. $s \in L(\#_{\gamma+1}^1(0))$ for $G_A$. $G_1^\gamma$ is similar to the game $G_0^\gamma$. The moves of $G_1^\gamma$ and $G_0^\gamma$ are the same with one exception: Once II plays $\langle \hat{t}_i^1, t_i^1\rangle = \langle 0, t_i^1\rangle$, in $G_0^\gamma$ I and II respectively play integer moves $x(2i)$ and $x(2i+1)$, whereas, in $G_1^\gamma$ I and II respectively play ordinals $\lambda_{2i}^0$ and $\lambda_{2i+1}^0$ with their respective integer moves $x(2i)$ and $x(2i+1)$. If II plays $\hat{t}_k^1 = 0$ for all $k \in \omega$, then the play of $G_1^\gamma$ is

$$\begin{array}{llllllllllll} \mathrm{I} & T_0^1 & & x(0),\lambda_0^0 & & T_1^1 & & x(2),\lambda_2^0 & & \cdots\ T_n^1 & & x(2n),\lambda_{2n}^0 \quad \cdots \\ \mathrm{II} & & \langle 0,t_0^1\rangle & & x(1),\lambda_1^0 & & \langle 0,t_1^1\rangle & & x(3),\lambda_3^0 & \quad\ \ \langle 0,t_n^1\rangle & & x(2n{+}1),\lambda_{2n+1}^0 \quad \cdots \end{array}$$

If II plays $\hat{t}^1_{k(1)} = 1$ and $\hat{t}^2_j = 0$ for all $j \in \omega$, then the play of $G^\gamma_1$ is

$$\begin{array}{lllllllll}
\mathrm{I} & T^1_0 & & x(0),\lambda^0_0 & & T^1_1 & & x(2),\lambda^0_2 & \ldots \\
\mathrm{II} & & \langle 0,t^1_0\rangle & & x(1),\lambda^0_1 & & \langle 0,t^1_1\rangle & & x(3),\lambda^0_3
\end{array}$$

$$\begin{array}{lllllll}
\ldots & T^1_{k(1)-1} & & x(2k(1)-2),\lambda^0_{2k(1)-2} & & T^1_{k(1)} & \\
 & & \langle 0,t^1_{k(1)-1}\rangle & & x(2k(1)-1),\lambda^0_{2k(1)-1} & & \langle 1,-\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^2_0 & & x(2k(1)),\lambda^1_0 & & T^2_1 & & x(2k(1)+2),\lambda^1_2 & & \ldots. \\
 & \langle 0,t^2_0\rangle & & x(2k(1)+1),\lambda^1_1 & & \langle 0,t^2_1\rangle & & x(2k(1)+3),\lambda^1_3 &
\end{array}$$

In general, if II plays $\hat{t}^j_{k(j)} = 1$ for $j \leq \delta$ and $\hat{t}^{\delta+1}_j = 0$ for all $j \in \omega$, then the play of $G^\gamma_1$ is

$$\begin{array}{lllllllll}
\mathrm{I} & T^1_0 & & x(0),\lambda^0_0 & & T^1_1 & & x(2),\lambda^0_2 & \ldots \\
\mathrm{II} & & \langle 0,t^1_0\rangle & & x(1),\lambda^0_1 & & \langle 0,t^1_1\rangle & & x(3),\lambda^0_3
\end{array}$$

$$\begin{array}{lllllll}
\ldots & T^1_{k(1)-1} & & x(2k(1)-2),\lambda^0_{2k(1)-2} & & T^1_{k(1)} & \\
 & & \langle 0,t^1_{k(1)-1}\rangle & & x(2k(1)-1),\lambda^0_{2k(1)-1} & & \langle 1,-\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^2_0 & & x(2k(1)),\lambda^1_0 & & T^2_1 & & x(2k(1)+2),\lambda^1_2 & & \ldots \\
 & \langle 0,t^2_0\rangle & & x(2k(1)+1),\lambda^1_1 & & \langle 0,t^2_1\rangle & & x(2k(1)+3),\lambda^1_3 &
\end{array}$$

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$$\begin{array}{lllllll}
\ldots & T^\delta_{k(\delta)-1} & & x(2k(1)\cdots+2k(\delta)-2),\lambda^{\delta-1}_{2k(\delta)-2} & & T^\delta_{k(\delta)} & \\
 & & \langle 0,t^\delta_{k(\delta)-1}\rangle & & x(2k(1)+\cdots+2k(\delta)-1),\lambda^{\delta-1}_{2k(\delta)-1} & & \langle 1,-\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^{\delta+1}_0 & & x(2k(1)+\cdots+2k(\delta)),\lambda^\delta_0 & & T^{\delta+1}_1 & & x(2k(1)+\cdots+2k(\delta)+2),\lambda^\delta_2 & & \ldots\ldots\ldots\ldots \\
 & \langle 0,t^{\delta+1}_0\rangle & & x(2k(1)+\cdots+2k(\delta)+1),\lambda^\delta_1 & & \langle 0,t^{\delta+1}_1\rangle & & x(2k(1)+\cdots+2k(\delta)+3),\lambda^\delta_3 &
\end{array}$$

$$\begin{array}{llllllllll}
\ldots & T^{\delta+1}_i & & x(2k(1)+\cdots+2k(\delta)+2i),\lambda^\delta_{2i} & & T^{\delta+1}_{i+1} & & x(2k(1)+\cdots+2k(\delta)+2i+2),\lambda^\delta_{2i+2} & & \ldots. \\
 & & \langle 0,t^{\delta+1}_i\rangle & & x(2k(1)+\cdots+2k(\delta)+2i+1),\lambda^\delta_{2i+1} & & \langle 0,t^{\delta+1}_{i+1}\rangle & & x(2k(1)+\cdots+2k(\delta)+2i+3),\lambda^\delta_{2i+3} &
\end{array}$$

Whenever $\hat{t}^\delta_{k(\delta)} = 1$ for $1 \leq \delta \leq \gamma$, the play of $G^\gamma_1$ is

$$\begin{array}{lllllllll}
\mathrm{I} & T^1_0 & & x(0),\lambda^0_0 & & T^1_1 & & x(2),\lambda^0_2 & \ldots \\
\mathrm{II} & & \langle 0,t^1_0\rangle & & x(1),\lambda^0_1 & & \langle 0,t^1_1\rangle & & x(3),\lambda^0_3
\end{array}$$

$$\begin{array}{lllllll}
\ldots & T^1_{k(1)-1} & & x(2k(1)-2),\lambda^0_{2k(1)-2} & & T^1_{k(1)} & \\
 & & \langle 0,t^1_{k(1)-1}\rangle & & x(2k(1)-1),\lambda^0_{2k(1)-1} & & \langle 1,-\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^2_0 & & x(2k(1)),\lambda^1_0 & & T^2_1 & & x(2k(1)+2),\lambda^1_2 & & \ldots \\
 & \langle 0,t^2_0\rangle & & x(2k(1)+1),\lambda^1_1 & & \langle 0,t^2_1\rangle & & x(2k(1)+3),\lambda^1_3 &
\end{array}$$

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$$\begin{array}{lllllll}
\ldots & T^\delta_{k(\delta)-1} & & x(2k(1)\cdots+2k(\delta)-2),\lambda^{\delta-1}_{2k(\delta)-2} & & T^\delta_{k(\delta)} & \\
 & & \langle 0,t^\delta_{k(\delta)-1}\rangle & & x(2k(1)+\cdots+2k(\delta)-1),\lambda^{\delta-1}_{2k(\delta)-1} & & \langle 1,-\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^{\delta+1}_0 & & x(2k(1)+\cdots+2k(\delta)),\lambda^\delta_0 & & T^{\delta+1}_1 & & x(2k(1)+\cdots+2k(\delta)+2),\lambda^\delta_2 & & \ldots \\
 & \langle 0,t^{\delta+1}_0\rangle & & x(2k(1)+\cdots+2k(\delta)+1),\lambda^\delta_1 & & \langle 0,t^{\delta+1}_1\rangle & & x(2k(1)+\cdots+2k(\delta)+3),\lambda^\delta_3 &
\end{array}$$

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$$\begin{array}{llllll} \ldots\ T^\gamma_{k(\gamma)-1} & & x(2k(1)+\cdots+2k(\gamma)-2),\lambda^{\gamma-1}_{2k(\gamma)-2} & & T^\gamma_{k(\gamma)} & \\ & \langle 0,t^\gamma_{k(\gamma)-1}\rangle & & x(2k(1)+\cdots+2k(\gamma)-1),\lambda^{\gamma-1}_{2k(\gamma)-1} & & \langle 1,-\rangle \end{array}$$

$$\begin{array}{lll} x(2k(0)+2k(1)+\cdots+2k(\gamma)),\xi_0 & x(2k(0)+2k(1)+\cdots+2k(\gamma)+2),\xi_2 & \\ \quad x(2k(0)+2k(1)+\cdots+2k(\gamma)+1),\xi_1 & \quad x(2k(0)+2k(1)+\cdots+2k(\gamma)+3),\xi_3 & \ldots\ldots\ldots\ldots\ldots\ldots\ldots . \end{array}$$

Any $\lambda^0_i$'s played must be properly ordered with respect to $\langle D_\alpha | \alpha < \omega \cdot m\rangle$ using $\langle \omega_{i+1}^{L(\#^1_{\gamma+1}(0))} | i < m\rangle$. Otherwise, the moves of $G^\gamma_1$ must satisfy all of the conditions required of the moves of $G^\gamma_0$. In particular, the $\lambda^\delta_i$'s are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (k(\delta)+1)\rangle$ using

$$\langle \omega_{i+1}^{L(\#^1_{\gamma-\delta+1}(T^1_{k(1)},T^2_{k(2)},\ldots,T^\delta_{k(\delta)}))} | i \le k(\delta)\rangle,$$

and any $\xi_i$'s played must be properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(\gamma)+1)\rangle \text{ using } \langle \omega_{i+1}^{L(\#^1_1(T^1_{k(1)},T^2_{k(2)},\ldots,T^\gamma_{k(\gamma)}))} | i \le k(\gamma)\rangle.$$

If $\hat{t}^j_{k(j)} = 1$ for $1 \le j < \delta$, then $G^\gamma_0$ contains a sequence

$$\langle (T^\delta_n; \langle \hat{t}^\delta_n, t^\delta_n\rangle) | \forall j < n\ \hat{t}^\delta_j = 0\rangle$$

of Borel auxiliary moves which is related to $B_\delta$ via $R_\delta$, and each

$$T^\delta_n \in L(\langle T^j_{k(j)} | 1 \le j < \delta\rangle)[\#^1_{\gamma+1-\delta}].$$

Player I wins $G^\gamma_1$ iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. $G^\gamma_1$ is an open game and therefore we define, for each ordinal $\alpha$, $P_\alpha$ as the set of positions with ordinal $\alpha$ and let $P = \bigcup_{\alpha \in \mathrm{ON}} P_\alpha$. If $p$ is a legal position in $G^\gamma_1$, let $\ell_p$ denote the set of legal positions in $G^\gamma_1$ consistent with $p$. The set $\ell$ of legal positions for $G^\gamma_1$ is in $L[\#^1_\gamma]$.

Use $\langle P_\alpha | \alpha \in \mathrm{ON}\rangle$ and Lemma 0.14 to define a wellordering $\prec$ of $\ell$ and the canonical w.s. $s^\gamma_1$ for $G^\gamma_1$ so that Lemma 2.3.1 (below) and the following hold: $s^\gamma_1$ is in $L[\#^1_\gamma]$, and if $p$ is a legal position in $G^\gamma_1$, then $s^\gamma_1 | \ell_p$ is a w.s. for

$(G^\gamma_1)_p$ and is definable in any inner model of ZF in which $\prec|\ell_p$ is definable.

**Lemma 2.3.1.** Let $p$ be a legal position in $G^\gamma_1$. Recall $\#^1_0(U) = U$ for any $U$. Then $s^\gamma_1|\ell_p$ is a w.s. for $(G^\gamma_1)_p$ and has the following property:

iii.) If $p$ includes the move $\langle \hat{t}^j_{k(j)}, t^j_{k(j)}\rangle = \langle 1, -\rangle$ for $1 \leq j \leq \delta$ and

$$\vec{T}_\delta = \langle T^j_{k(j)} | 1 \leq j \leq \delta\rangle,$$

then $s^\gamma_1|\ell_p$ is definable in $L(\vec{T}_\delta)[\#^1_{\gamma-\delta}]$ from $\langle \omega^{L(\#^1_{\gamma+1-\delta}(\vec{T}_\delta))}_{i+1} | i \leq k(\delta)\rangle$.

In particular,

iv.) if $p$ includes the move $\langle \hat{t}^1_{k(1)}, t^1_{k(1)}\rangle = \langle 1, -\rangle$, then $s^\gamma_1|\ell_p$ is definable in $L(T^1_{k(1)})[\#^1_{\gamma-1}]$ from $\langle \omega^{L(\#^1_\gamma(T^1_{k(1)}))}_{i+1} | i \leq k(1)\rangle$.

By Lemma 2.3.1 and Induction Hypothesis, we have the following:

**Lemma 2.3.2.** Suppose

$$p_k = (T^1_0; \langle 0, t^1_0\rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; T^1_1; \langle 0, t^1_1\rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; \ldots$$
$$\ldots; T^1_{k-1}; \langle 0, t^1_{k-1}\rangle; x(2k-2), \lambda^0_{2k-2}; x(2k-1), \lambda^0_{2k-1}; T^1_k; \langle 1, -\rangle)$$

is a legal position of $G^\gamma_1$. Let $B$ be the set witnessed to be $((\gamma-1) * \Pi^0_1)^*_+$ by $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, \ldots, B_2$, $\langle A_\alpha | \alpha < \omega^2\rangle$, and $k$. If $B_1(x, k)$, then

$$x \in B(T^1_k; \vec{t}^1, \bar{x}(2k)) \Leftrightarrow x \in A(T^1_k; \vec{t}^1, \bar{x}(2k)).$$

Moreover, the w.s. $s^\gamma_1|\ell_{p_k}$ can be integrated so as to obtain a w.s. $s' \in L(\#^1_\gamma(T^1_k))$ for $B(T^1_k; t^1_0, t^1_1, \ldots, t^1_{k-1}, \bar{x}(2k))$ so that the following hold:

v.) $s'$ is a w.s. of the player for whom $s^\gamma_1$ is a w.s.

vi.) If $s'$ is a w.s. for I, $p$ is a position consistent with $s'$, and the moves in $p$ of player II are consistent with $\bar{x}(2k+1)$ and $\langle t^1_j | j < k\rangle$, then $p \in$

$T^1_k$. Therefore, if $s'$ is a w.s. for I and $x$ is a play consistent with $s'$, $x \in B(t^1_0, t^1_1, t^1_2, ..., t^1_{k-1}, \bar{x}(2k))$. If in addition $B_1(x, k)$, then $x \in A(\vec{t}^1, \bar{x}(2k))$.

vii.) If $p$ is a (legal) position of the game $A(T^1_k)$ consistent with $s'$ such that the moves in $p$ of the player for whom $s'$ is not a w.s. are consistent with $\vec{t}^1$ and $p$ is consistent with $s'$, then $p$ is consistent with $\vec{t}^1$.

We integrate $s^\gamma_1$ similarly to the manner in which $s^\gamma_0$ is integrated in Theorem 2.2, except that $s^\gamma_1$ is integrated with respect to the $\lambda^0_i$'s using the uncountable set $C^\gamma_1$ of indiscernibles for $L[\#^1_\gamma]$. We use the following lemma to integrate $s^\gamma_1$ with respect to the $\lambda^0_i$'s.

**Lemma 2.3.3.** Let

$$p = (T^1_0; \langle 0, t^1_0 \rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; T^1_1; \langle 0, t^1_1 \rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; ...$$
$$...; T^1_{n-1}; \langle 0, t^1_{n-1} \rangle; x(2n-2), \lambda^0_{2n-2}; x(2n-1), \lambda^0_{2n-1}; T^1_n; \langle \hat{t}^1_n, t^1_n \rangle)$$

and

$$p' = (U^1_0; \langle 0, u^1_0 \rangle; x'(0), \tilde{\lambda}^0_0; x'(1), \tilde{\lambda}^0_1; U^1_1; \langle 0, u^1_1 \rangle; x'(2), \tilde{\lambda}^0_2; x'(3), \tilde{\lambda}^0_3; ...$$
$$...; U^1_{n-1}; \langle 0, u^1_{n-1} \rangle; x'(2n-2), \tilde{\lambda}^0_{2n-2}; x'(2n-1), \tilde{\lambda}^0_{2n-1}; U^1_n; \langle \hat{u}^1_n, u^1_n \rangle)$$

be legal positions of $G^\gamma_1$ consistent with $s^\gamma_1$.

(1) If $s^\gamma_1$ is a w.s. for I, $\lambda^0_{2i-1}$ and $\tilde{\lambda}^0_{2i-1}$ are elements of $C^\gamma_1$ for $i \leq n$, and both the Borel auxiliary moves and the integer moves of player II are the same for $p$ and $p'$ (i.e. $x(2i-1) = x'(2i-1)$, $t^1_i = u^1_i$, and $\langle \hat{t}^1_n, t^1_n \rangle = \langle \hat{u}^1_n, u^1_n \rangle$), then both the Borel auxiliary moves and integer moves of player I are the same for both $p$ and $p'$ (i.e. $x(2i) = x'(2i)$ and $T^1_i = U^1_i$).

(2) If $s_1^\gamma$ is a w.s. for II, $\lambda_{2i}^0$ and $\tilde\lambda_{2i}^0$ are elements of $C_1^\gamma$ for $i < n$, and both the Borel auxiliary moves and the integer moves of player I are the same for $p$ and $p'$ (i.e. $x(2i) = x'(2i)$ and $T_i^1 = U_i^1$), then both the Borel auxiliary moves and integer moves of player II are the same for both $p$ and $p'$ (i.e. $x(2i-1) = \tilde x(2i-1)$, $t_i^1 = u_i^1$, and $\langle \hat t_n^1, t_n^1\rangle = \langle \hat u_n^1, u_n^1\rangle$).

Now we integrate $s_1^\gamma$ and obtain the w.s. $s$ for $G_A$.

**Claim I:** Player I has a w.s. for $G_A$ if he has one for $G_1^\gamma$.

Assume $\langle\ \rangle \in P$ so that $s_1^\gamma \in L[\#_\gamma^1]$ is a w.s. for I in $G_1^\gamma$. We use $s_1^\gamma$ to define a w.s. $s$ for I in $G_A$. Let

$$T_0^1 = s_1^\gamma(\langle\ \rangle),\ \langle \hat t_0^1, t_0^1\rangle = \langle 1, -\rangle, \text{ and } p_0 = (T_0^1; \langle 1, -\rangle).$$

By Lemma 2.3.2(v), obtain a w.s. $s_1$ for $A(T_0^1)$ by integrating the w.s. $s_1^\gamma|\ell_{p_0}$ for $(G_1^\gamma)_{p_0}$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar x(j), 0)$. If $R_1(\bar x(i), 0)$ holds at every position, then $x \in A(T_0^1)$ so that $x \in A$ by Lemma 2.3.2(vi).

Suppose we reach a position such that $\neg R_1(\bar x(i_0), 0)$ and $\forall j < i_0 R_1(\bar x(j), 0)$. We have defined

$$s(x(1); x(3); ...; x(2j-1)) = x(2j) \text{ for } 2j < i_0. \qquad \text{(viii)}$$

Since $i_0$ is odd by (ii), let $\langle \hat t_0^1, t_0^1\rangle = \langle 0, \bar x(i_0)\rangle$ and $p_0' = (T_0^1; \langle 0, t_0^1\rangle)$. Let $(x(0), \lambda_0^0)$ be $s_1^\gamma(p_0')$ and $s(\ ) = x(0)$. Choose

$$\tilde\lambda_1^0 \in C_1^\gamma \text{ and } p_1 = p_0' * (x(0), \tilde\lambda_0^0; x(1), \tilde\lambda_1^0; T_1^1; \langle 1, -\rangle)$$

so that $p_1$ is consistent with $s_1^\gamma$. By Lemma 2.3.2(v), obtain a w.s. $s_1$ for

$A(T^1_1; t^1_0, \bar{x}(2))$ by integrating the w.s. $s^\gamma_1|\ell_{p_1}$ for $(G^\gamma_1)_{p_1}$, and let $s(p) = s_1(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 1)$. By (vii), this definition of $s$ is consistent with (viii). If $R_1(\bar{x}(i), 1)$ holds at every position, then $x \in A(T^1_1; t^1_0, \bar{x}(2))$ so that $x \in A$ by Lemma 2.3.2. If we reach a position such that $\neg R_1(\bar{x}(i_1), 1)$ and $\forall j < i_1 R_1(\bar{x}(j), 1)$, then continue to define $s$ by integrating $s^\gamma_1$ in the same manner as above.

In general, suppose we reach a position such that

$$\neg R_1(\bar{x}(i_j), j) \text{ and } \forall i < i_j R_1(\bar{x}(i), j)$$

for $j = 1, 2, 3, ..., k-1$. Choose $\lambda^0_1, \lambda^0_3, ..., \lambda^0_{2k-1} \in C^\gamma_1$ and

$$p_k = (T^1_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; T^1_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; ...$$
$$...; T^1_{k-1}; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda^0_{2k-2}; x(2k-1), \lambda^0_{2k-1}; T^1_k; \langle 1, -\rangle)$$

so that $p_k$ is consistent with $s^\gamma_1$. ($\lambda^0_1 \in C^\gamma_1$ may be different than the ordinal auxiliary move $\tilde{\lambda}^0_1$ of $p_1$, but by Lemma 2.3.3, the integer moves and Borel auxiliary moves of $p_1$ are included in $p_k$.) By Lemma 2.3.2(v), obtain a w.s. $s_k \in L(\#^1_\gamma(T^1_k))$ for

$$A(T^1_k; t^1_0, t^1_1, ..., t^1_{k-1}, \bar{x}(2k))$$

by integrating the w.s. $s^\gamma_1|\ell_{p_k}$ for $(G^\gamma_1)_{p_k}$, and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), k)$. If we reach a position $\bar{x}(i_k)$ such that $\neg R_1(\bar{x}(i_k), k)$, then the position

$$p'_k = (T^1_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; T^1_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; ...$$
$$...; T^1_{k-1}; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda^0_{2k-2}; x(2k-1), \lambda^0_{2k-1}; T^1_k; \langle 0, \bar{x}(i_k)\rangle)$$

is consistent with $s_1^\gamma$.

**Claim:** The strategy $s$ of player I is a w.s. for $G_A$.

Let $x$ be a play of $G_A$ consistent with $s$. First assume there is a least $k$ such that $B_1(x,k)$. Then for each $j < k$, there is a least $i_j$ such that $\neg R_1(\bar{x}(i_j), j)$. By the definition of $s$, there exist

$$T_0^1, T_1^1, T_2^1, ..., T_k^1 \text{ and } \lambda_1^0, \lambda_3^0, \lambda_5^0, ..., \lambda_{2k-1}^0 \in C_1^\gamma$$

such that the position

$$p_k = (T_0^1; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; T_1^1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda_2^0; x(3), \lambda_3^0; ...$$
$$...; T_{k-1}^1; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda_{2k-2}^0; x(2k-1), \lambda_{2k-1}^0; T_k^1; \langle 1, -\rangle)$$

is consistent with $s_1^\gamma$. We obtained the w.s. $s_k$ for

$$A(T_k^1; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k))$$

by integrating the w.s. $s_1^\gamma | \ell_{p_k}$ for $(G_1^\gamma)_{p_k}$ and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), k)$. Since $B_1(x,k)$, $\forall j R_1(\bar{x}(j), k)$. Therefore, $x$ is a play consistent with $s_k$ so that

$$x \in A(T_k^1; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k)).$$

Clearly, $x$ is consistent with each $\bar{x}(i_j)$ and by Lemma 2.3.2(vi), $x \in T_k^1$. Thus, $x \in A$ if $\exists k B_1(x,k)$.

Now assume $\neg B_1(x,k)$ for all $k$. Then for each $k$, there is a least $i_k$ such that $\neg R_1(\bar{x}(i_k), k)$. By the definition of $s$, for each $k$, there exists $\lambda_1^0, \lambda_3^0, \lambda_5^0, ..., \lambda_{2k-1}^0 \in C_1^\gamma$ such that the position

$$p'_k = (T_0^1; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; T_1^1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda_2^0; x(3), \lambda_3^0; ...$$

$$...; T^1_{k-1}; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda^0_{2k-2}; x(2k-1), \lambda^0_{2k-1}; T^1_k; \langle 0, \bar{x}(i_k)\rangle)$$

is consistent with $s^\gamma_1$. Show $x \in D^*_{\omega \cdot m} = D$ by using Lemma 2.3.3, the positions $p'_k$, and the elements of $C^\gamma_1$ to integrate $s^\gamma_1$ with respect to the $\lambda^0_{2i+1}$'s. Since $x \in D$ and $\forall k \neg B_1(x, k)$, $x \in A$. Thus, in either case, $x$ is a win for I, and $s$ is a w.s. of I.

**Claim II:** Player II has a w.s. for $G_A$ if he has one for $G^\gamma_1$.

Again, we use the uncountable set $C^\gamma_1$ of indiscernibles for $L[\#^1_\gamma]$ to integrate $s^\gamma_1$ with respect to $\lambda^0_i$'s and otherwise we integrate $s^\gamma_1$ similarly to the manner in which $s^\gamma_0$ is integrated in Theorem 2.2.

Assume $\langle\ \rangle \notin P$. Let

$$T^1_0 = \{\text{positions } t \text{ in } G_A | \forall T' \in L[\#^1_\gamma]\ \langle 0, t\rangle \neq s^\gamma_1(T')\}.$$

Then $\forall t \in T^1_0 \forall T' \in L[\#^1_\gamma]\ \langle 0, t\rangle \neq s^\gamma_1(T')$. (ix)

x.) If $(T'; \langle 0, t\rangle)$ is a legal position of $G^\gamma_1$, then for each ordinal $\alpha$, $P_\alpha \cap \ell_{(T'; \langle 0, t\rangle)}$ is definable in $L[\#^1_\gamma]$ from $\langle \omega^{L(\#^1_{\gamma+1}(0))}_{i+1} | i < m\rangle$.

Therefore, $T^1_0 \in L[\#^1_\gamma]$. Also, by (ix), $\langle 1, -\rangle = s^\gamma_1(T^1_0)$. Let $p_0 = (T^1_0, \langle 1, -\rangle)$. By Lemma 2.3.2(v), obtain a w.s. $s_0$ for $A(T^1_0)$ by integrating the w.s. $s^\gamma_1 | \ell_{p_0}$ for $(G^\gamma_1)_{p_0}$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i\ \bar{x}(j) \in T^1_0$. If $\bar{x}(i) \in T^1_0$ holds at every position, then $x \notin A(T^1_0)$ so that $x \notin A$.

Suppose we reach a position such that $\bar{x}(i_0) \notin T^1_0$ and $\forall j < i_0\ \bar{x}(j) \in T^1_0$. We have defined

$$s(x(0);x(2);...;x(2j)) = x(2j+1) \text{ for } 2j < i_0. \tag{xi}$$

Since $\bar{x}(i_0) \notin T^1_0$, there exists $\tilde{T}^1_0 \in L[\#^1_\gamma]$ such that $\langle 0, \bar{x}(i_0)\rangle = s^\gamma_1(\tilde{T}^1_0)$.

Let $\langle \hat{t}^1_0, t^1_0\rangle = \langle 0, \bar{x}(i_0)\rangle$ and let $p'_0 = (\tilde{T}^1_0; \langle 0, t^1_0\rangle; x(0), \lambda^0_0)$ be a position consitent with $s^\gamma_1$ in which $\lambda^0_0 \in C^1_\gamma$. Let $(x(1), \lambda^0_1) = s^\gamma_1(p'_1)$ and define $s(x(0))$ to be $x(1)$. Let

$$T^1_1 = \{\text{positions } t \text{ in } G_A(\bar{x}(2), t^1_0) | \, \forall T' \in L[\#^1_\gamma] \, \langle 0, t\rangle \neq s^\gamma_1(p'_0 * (x(1), \lambda^0_1; T'))\}.$$

Then $T^1_1 \in L[\#^1_\gamma]$ and $\langle 1, -\rangle = s^\gamma_1(p'_0 * (x(1), \lambda^0_1; T^1_1))$. Let

$$p_1 = p'_0 * (x(1), \lambda^0_1; T^1_1; \langle 1, -\rangle).$$

By Lemma 2.3.2(v), obtain a w.s. $s_1$ for $A(T^1_1; t^1_0, \bar{x}(2))$ by integrating the w.s. $s^\gamma_1 | \ell_{p_1}$ for $(G^\gamma_1)_{p_1}$, and let $s(p) = s_1(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i R_1(\bar{x}(j), 1)$. By Lemma 2.3.2(vi), this definition of $s$ is consistent with (xi). If $\bar{x}(i) \in T^1_1$ holds at every position, then $x \notin A(T^1_1; t^1_0, \bar{x}(2))$ so that $x \notin A$. If we reach a position such that $\bar{x}(i_1) \notin T^1_1$, then continue to define $s$ by integrating $s^\gamma_1$ in the same manner as above.

In general, suppose we reach a position at which the following hold:

$$t^1_j = \bar{x}(i_j) \notin T^1_j \text{ and } \forall i < i_j \; \bar{x}(i) \in T^1_j \text{ for } j = 1, 2, 3, ..., k-1,$$

and $\bar{x}(2k-1)$ is consistent with $s$. Also, assume $\tilde{T}^1_0, \tilde{T}^1_1, \tilde{T}^1_2, ..., \tilde{T}^1_{k-1}$ satisfy the following: For each $j \leq k$, there exists a sequence $\vec{\lambda}_j = \langle \lambda^0_{2i} | i < j\rangle$ of elements from $C^\gamma_1$ and a position

$$p'_{j-1} = (\tilde{T}^1_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda^0_0; x(1), \lambda^0_1; \tilde{T}^1_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda^0_2; x(3), \lambda^0_3; ... \\ ...; \tilde{T}^1_{j-1}; \langle 0, \bar{x}(i_{j-1})\rangle; x(2j-2), \lambda^0_{2j-2})$$

which is consistent with $s_1^\gamma$. ($\vec{\lambda}_j$ is not necessarily a subsequence of $\vec{\lambda}_{j+1}$.) Define $s(x(0);x(2);...;x(2k-2)) = x(2k-1)$ to be such that $p'_{k-1} * (x(2k-1), \lambda^0_{2k-1})$ is consistent with $s_1^\gamma$ for some $\lambda^0_{2k-1}$. Let

$$T_k^1 = \{\text{positions } t \text{ in } G_A \text{ consistent with } t_0^1, t_1^1, t_2^1, ..., t_{k-1}^1, \text{ and } \bar{x}(2k) | \\ \forall T' \in L[\#_\gamma^1]\ \langle 0, t\rangle \neq s_1^\gamma(p'_{k-1} * (x(2k-1), \lambda^0_{2k-1}; T'))\}.$$

Then $T_k^1 \in L[\#_\gamma^1]$ and $\langle 1, -\rangle = s_1^\gamma(p'_{k-1} * (x(2k-1), \lambda^0_{2k-1}; T_k^1))$. Let

$$p_k = p'_{k-1} * (x(2k-1), \lambda^0_{2k-1}; T_k^1; \langle 1, -\rangle).$$

By Lemma 2.3.2(v), obtain a w.s. $s_k \in L(\#_\gamma^1(T_k^1))$ for $A(T_k^1; t_0^1, t_1^1, ..., t_{k-1}^1, \bar{x}(2k))$ by integrating the w.s. $s_1^\gamma | \ell_{p_k}$ for $(G_1^\gamma)_{p_k}$, and let

$$s(p) = s_k(p)$$

for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i\ \bar{x}(j) \in T_k^1$.

**Claim:** The strategy $s$ of player II is a w.s. for $G_A$.

Let $x$ be a play of $G_A$ consistent with $s$. First consider the case in which there is a least $k$ such that $\forall i\ \bar{x}(i) \in T_k^1$. Then for each $j < k$, there is a least $i_j$ such that $\bar{x}(i_j) \notin T_j^1$. By the definition of $x(2j+1) = s(x(0); x(2); ...; x(2j))$ for $j < k$ and by the definition of the $T_i^1$'s, there exist

$$\tilde{T}_0^1, \tilde{T}_1^1, \tilde{T}_2^1, ..., \tilde{T}_{k-1}^1 \in L[\#_\gamma^1];\ \lambda_0^0, \lambda_2^0, \lambda_4^0, ..., \lambda^0_{2k-2} \in C_1^\gamma; \text{ and}$$

$$p_k = (\tilde{T}_0^1; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; \tilde{T}_1^1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda_2^0; x(3), \lambda_3^0; ... \\ ...; \tilde{T}_{k-1}^1; \langle 0, \bar{x}(i_{k-1})\rangle; x(2k-2), \lambda^0_{2k-2}; x(2k-1), \lambda^0_{2k-1}; T_k^1; \langle 1, -\rangle)$$

such that $p_k$ is consistent with $s_1^\gamma$. We obtained the w.s. $s_k$ for

$$A(T_k^1; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k))$$

by integrating the w.s. $s_1^\gamma|\ell_{p_k}$ for $(G_1^\gamma)_{p_k}$ and let $s(p) = s_k(p)$ for any position $p = (x(0); x(1); ...; x(i-1))$ such that $\forall j \leq i \; \bar{x}(j) \in T_k^1$. Since we are assuming $\forall j \; \bar{x}(j) \in T_k^1$, $x$ is a play consistent with $s_k$ so that $x \notin A(T_k^1; \bar{x}(i_0), \bar{x}(i_1), ..., \bar{x}(i_{k-1}), \bar{x}(2k))$. Therefore, $x \notin A$.

Now assume for each $k$, there is a least $i_k$ such that $\bar{x}(i_k) \notin T_k^1$. By the definition of $x(2j+1) = s(x(0); x(2); ...; x(2j))$ for $j < k$ and by the definition of the $T_i^1$'s, there exist for each $j$, $\tilde{T}_j^1 \in L[\#_\gamma^1]$ such that the following hold:

For each $k$, there exists $\lambda_0^0, \lambda_2^0, \lambda_4^0, ..., \lambda_{2k}^0 \in C_1^\gamma$ such that the position

$$p'_k = (\tilde{T}_0^1; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda_0^0; x(1), \lambda_1^0; \tilde{T}_1^1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda_2^0; x(3), \lambda_3^0; ...$$
$$...; \tilde{T}_k^1; \langle 0, \bar{x}(i_k)\rangle; x(2k), \lambda_{2k}^0)$$

is consistent with $s_1^\gamma$. Since each such position is consistent with $s_1^\gamma$, $\forall k \neg R_1(\bar{x}(i_k), k)$ so that $\forall k \neg B_1(x, k)$. Show $x \notin D$ by using Lemma 2.3.3, the positions $p'_k$, and the elements of $C_1^\gamma$ to integrate $s_1^\gamma$ with respect to the $\lambda_{2i}^0$'s. Since $\forall k \neg B(x, k)$ and $x \notin D$, $x$ is a win for II. Consequently, $s$ is a w.s. in $G_A$ of the player for whom $s_1^\gamma$ is w.s. ∎

**Definition 2.3.** Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi_1^0$, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $k \in \omega$ strongly witness $A \in (\gamma * \Pi_1^0)_+^*$. Then we refer to the auxiliary game $G_1^\gamma$ described in the Proof of Theorem 2.3 as *the $G_1^\gamma$ auxiliary game determined by $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi_1^0$, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $k \in \omega$* if $m = k+1$ and $D_\alpha = A_\alpha$ for $\alpha < \omega \cdot m$.

Suppose $\vec{U} = \langle U_i | i < \beta\rangle$ and $\vec{u} = \langle u_i | i < \gamma\rangle$ respectively are a finite sequence of I-imposed subgames of $G_A$ and a sequence of legal positions of $G_A$. Then *the* $G_1^\gamma(\vec{U};\vec{u})$ *auxiliary game determined by*

$$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1,\ \langle A_\alpha \mid \alpha < \omega^2\rangle,\ \textit{and}\ k \in \omega$$

is the game in which player I wins iff a position is reached at which II cannot make a (legal) move, which has exactly the same moves as $G_1^\gamma$, and these moves are subject to the following conditions:

i.) The sequence $\langle (T_n^\delta; \langle \hat{t}_n^\delta, t_n^\delta\rangle)|\forall j < n\ \hat{t}_j^\delta = 0\rangle$ of Borel auxiliary moves and the $\Pi_1^0$ set $B_\delta$ are related via $R_{B_\delta}$.

ii.) Each $\bar{x}(i) \in \bigcap_{j<\beta} U_j$ and each $\bar{x}(i)$ must be consistent with every $u_j$.

iii.) $T_i^\delta \in L(\vec{U}, \vec{T}_{\delta-1})[\#^1_{\gamma+1-\delta}]$. Recall $\vec{T}_{\delta-1} = \langle T^j_{k(j)}|1 \leq j < \delta\rangle$, where $\hat{t}^j_{k(j)} = 1$.

iv.) Let $k(0) = k$. The $\lambda_i^\delta$'s are properly ordered with repect to $\langle A_\alpha|\alpha < \omega\cdot(k(\delta)+1)\rangle$ using $\langle \omega_{i+1}^{L(\#^1_{\gamma+1-\delta}(\vec{U},\vec{T}_\delta))}|i \leq k(\delta)\rangle$

v.) If $\hat{t}^\gamma_{k(\gamma)} = 1$, the $\xi_i$'s are properly ordered with respect to $\langle A_\alpha|\alpha < \omega\cdot(k(\gamma)+1)\rangle$ using $\langle \omega_{i+1}^{L(\#^1_1(\vec{U},\vec{T}))}|i \leq k(\gamma)\rangle$.

These conditions are analogous to the conditions for the moves of $G_1^\gamma$. The first is a condition which the moves of $G_1^\gamma$ also must satisfy. The others are derived by changing the conditions for the moves of $G_1^\gamma$ so that we obtain conditions which are consistent with $\vec{U}$ and $\vec{u}$. We refer to $G_1^\gamma(\vec{U};\vec{u})$ instead of the $G_1^\gamma(\vec{U};\vec{u})$ auxiliary game determined by $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1$, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $k \in \omega$ whenever $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1$, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $k \in \omega$

are clear from the context. Analogous to Theorem 2.3, we have the following:

**Corollary 2.3.1.** Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, $k$, $A$, $\vec{U}$, and $\vec{u}$ be as in Definition 2.3. Let $p$ be a legal position of a game $G^*$ such that the moves of $G^*$ following $p$ constitute a play of $G_1^\gamma(\vec{U};\vec{u})$. Suppose $\vec{U}$ has a wellordering which is definable in $L(\vec{U})$, $\#^1_{\gamma+1}(\vec{U})$ exists, and $s^*$ is a w.s. for $G^*$ such that $s^*|\ell_p \in L(\vec{U})[\#^1_\gamma]$. Then $s^*|\ell_p$ can be integrated so as to obtain a w.s. $s_p \in L(\#^1_{\gamma+1}(\vec{U}))$ for $A(\vec{U};\vec{u})$ such that the following hold:

i.) $s_p$ is a w.s. of the player for whom $s^*$ is a w.s.

ii.) If $s^*$ is a w.s. for I, $\hat{p}$ is a position consistent with $s_p$, and the moves in $\hat{p}$ of player II are consistent with $\vec{u}$, then $\hat{p} \in \bigcap_{i<\beta} U_i$. Therefore, if $s^*$ is a w.s. for I and $x$ is a play consistent with $s_p$, then $x \in A(\vec{u})$.

iii.) Let $\hat{p}$ be a position consistent with $s_p$ and with $\vec{U}$. If the moves in $\hat{p}$ of the player for whom $s_p$ is not a w.s. are consistent with $\vec{u}$, then $\hat{p}$ is consistent with $\vec{u}$. ∎