Now we prove the generalization of all Theorems $x.y$ (of this paper), where $y$ is odd and $x \neq 0$.

**Theorem 2.5.** If $0^{(\beta+1)\#^1_{\gamma+1}}$ exists (i.e. $L(0^{\beta\#^1_{\gamma+1}})[\#^1_\gamma]$ has indiscernibles), then $\mathrm{Det}(\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$.

**Proof:** Assume $L(0^{\beta\#^1_{\gamma+1}})[\#^1_\gamma]$ has an uncountable set of indiscernibles. Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi^0_1$, $C_1, C_2, C_3, ..., C_\beta \in \Sigma^0_1$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ strongly witness $\mathrm{A} \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$. Wlog $C_1 \subseteq C_2 \subseteq C_3 \subseteq ... \subseteq C_\beta$. Then for $1 \leq i \leq \beta$ and $1 \leq j \leq \gamma$, there exist $R_{C_i}$ and $R_{B_j}$ in $\Delta^0_1$ such that

i.) $B_j(x,n) \leftrightarrow \forall k R_{B_j}(\bar{x}(k), n)$;

ii.) $C_i(x,n) \leftrightarrow \exists k R_{C_i}(\bar{x}(k), n)$;

iii.) if $\neg R_{B_j}(\bar{x}(k), n)$ and $\forall k' < k R_{B_j}(\bar{x}(k'), n)$, then $k$ is odd; and

iv.) if $R_{C_i}(\bar{x}(k), n)$ and $\forall k' < k \neg R_{C_i}(\bar{x}(k'), n)$, then $k$ is odd.

We show that $G_A$ has a w.s. $s$. Conditions (iii) and (iv) help to simplify the proof.

We describe an open game $G^\gamma_{2\beta+1}$ which has a w.s. $s^\gamma_{2\beta+1} \in L(0^{\beta\#^1_{\gamma+1}})[\#^1_\gamma]$. We integrate $s^\gamma_{2\beta+1}$ to get the w.s. $s \in L(0^{(\beta+1)\#^1_{\gamma+1}})$ for $G_A$. $G^\gamma_{2\beta+1}$ is similar to the game $G^\gamma_{2\beta}$. The moves of $G^\gamma_{2\beta}$ and $G^\gamma_{2\beta+1}$ are the same with one exception: In $G^\gamma_{2\beta+1}$, I and II always play an ordinal auxiliary move $\lambda^0_i$ with every integer move $x(i)$ which does not follow the play of some $\langle \hat{t}^1_{k(1)}, t^1_{k(1)} \rangle = \langle 1, - \rangle$; whereas, in $G^\gamma_{2\beta}$, I and II never play an ordinal auxliary move $\lambda^0_i$ whenever $\hat{q}_\beta = 1$.

Whenever II plays $\hat{t}^j_{k(j)} = 1$ for $j \le \delta$ and $\hat{t}^{\delta+1}_j = 0$ for all $j \in \omega$, the play of $G^\gamma_{2\beta+1}$ is

$$\begin{array}{llllll}
\mathrm{I} & Q_\beta & & Q_{\beta-1} & & Q_{\beta-2} \quad \ldots \\
\mathrm{II} & & \langle q_\beta, q_\beta\rangle & & \langle \hat{q}_{\beta-1}, q_{\beta-1}\rangle & \quad \langle \hat{q}_{\beta-2}, q_{\beta-2}\rangle
\end{array}$$

$$\begin{array}{lllllllll}
\ldots\ Q_1 & & T^1_0 & & x(0),\lambda^0_0 & & T^1_1 & & x(2),\lambda^0_2 \quad \ldots \\
& \langle \hat{q}_1, q_1\rangle & & \langle 0, t^1_0\rangle & & x(1),\lambda^0_1 & & \langle 0, t^1_1\rangle & \quad x(3),\lambda^0_3
\end{array}$$

$$\begin{array}{llllll}
\ldots\ T^1_{k(1)-1} & & x(2k(1)-2),\lambda^0_{2k(1)-2} & & T^1_{k(1)} & \\
& \langle 0, t^1_{k(1)-1}\rangle & & x(2k(1)-1),\lambda^0_{2k(1)-1} & & \langle 1, -\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^2_0 & & x(2k(1)),\lambda^1_0 & & T^2_1 & & x(2k(1)+2),\lambda^1_2 & & \ldots \\
& \langle 0, t^2_0\rangle & & x(2k(1)+1),\lambda^1_1 & & \langle 0, t^2_1\rangle & & x(2k(1)+3),\lambda^1_3 &
\end{array}$$

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$$\begin{array}{llllll}
\ldots\ T^\delta_{k(\delta)-1} & & x(2k(1)\cdots+2k(\delta)-2),\lambda^{\delta-1}_{2k(\delta)-2} & & T^\delta_{k(\delta)} & \\
& \langle 0, t^\delta_{k(\delta)-1}\rangle & & x(2k(1)+\cdots+2k(\delta)-1),\lambda^{\delta-1}_{2k(\delta)-1} & & \langle 1, -\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^{\delta+1}_0 & & x(2k(1)+\cdots+2k(\delta)),\lambda^\delta_0 & & T^{\delta+1}_1 & & x(2k(1)+\cdots+2k(\delta)+2),\lambda^\delta_2 & & \ldots \\
& \langle 0, t^{\delta+1}_0\rangle & & x(2k(1)+\cdots+2k(\delta)+1),\lambda^\delta_1 & & \langle 0, t^{\delta+1}_1\rangle & & x(2k(1)+\cdots+2k(\delta)+3),\lambda^\delta_3 &
\end{array}$$

$$\begin{array}{lllll}
\ldots\ T^{\delta+1}_i & & x(2k(1)+\cdots+2k(\delta)+2i),\lambda^\delta_{2i} & & \ldots. \\
& \langle 0, t^{\delta+1}_i\rangle & & x(2k(1)+\cdots+2k(\delta)+2i+1),\lambda^\delta_{2i+1} &
\end{array}$$

If $\hat{t}^\delta_{k(\delta)} = 1$ for $1 \le \delta \le \gamma$, the play of $G^\gamma_{2\beta+1}$ is

$$\begin{array}{llllll}
\mathrm{I} & Q_\beta & & Q_{\beta-1} & & Q_{\beta-2} \quad \ldots \\
\mathrm{II} & & \langle \hat{q}_\beta, q_\beta\rangle & & \langle \hat{q}_{\beta-1}, q_{\beta-1}\rangle & \quad \langle \hat{q}_{\beta-2}, q_{\beta-2}\rangle
\end{array}$$

$$\begin{array}{lllllllll}
\ldots\ Q_1 & & T^1_0 & & x(0),\lambda^0_0 & & T^1_1 & & x(2),\lambda^0_2 \quad \ldots \\
& \langle \hat{q}_1, q_1\rangle & & \langle 0, t^1_0\rangle & & x(1),\lambda^0_1 & & \langle 0, t^1_1\rangle & \quad x(3),\lambda^0_3
\end{array}$$

$$\begin{array}{llllll}
\ldots\ T^1_{k(1)-1} & & x(2k(1)-2),\lambda^0_{2k(1)-2} & & T^1_{k(1)} & \\
& \langle 0, t^1_{k(1)-1}\rangle & & x(2k(1)-1),\lambda^0_{2k(1)-1} & & \langle 1, -\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^2_0 & & x(2k(1)),\lambda^1_0 & & T^2_1 & & x(2k(1)+2),\lambda^1_2 & & \ldots \\
& \langle 0, t^2_0\rangle & & x(2k(1)+1),\lambda^1_1 & & \langle 0, t^2_1\rangle & & x(2k(1)+3),\lambda^1_3 &
\end{array}$$

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$$\begin{array}{llllll}
\ldots\ T^\delta_{k(\delta)-1} & & x(2k(1)\cdots+2k(\delta)-2),\lambda^{\delta-1}_{2k(\delta)-2} & & T^\delta_{k(\delta)} & \\
& \langle 0, t^\delta_{k(\delta)-1}\rangle & & x(2k(1)+\cdots+2k(\delta)-1),\lambda^{\delta-1}_{2k(\delta)-1} & & \langle 1, -\rangle
\end{array}$$

$$\begin{array}{lllllllll}
T^{\delta+1}_0 & & x(2k(1)+\cdots+2k(\delta)),\lambda^\delta_0 & & T^{\delta+1}_1 & & x(2k(1)+\cdots+2k(\delta)+2),\lambda^\delta_2 & & \ldots \\
& \langle 0, t^{\delta+1}_0\rangle & & x(2k(1)+\cdots+2k(\delta)+1),\lambda^\delta_1 & & \langle 0, t^{\delta+1}_1\rangle & & x(2k(1)+\cdots+2k(\delta)+3),\lambda^\delta_3 &
\end{array}$$

.................................................................

$$\begin{array}{llllll}
\ldots\ T^\gamma_{k(\gamma)-1} & & x(2k(1)+\cdots+2k(\gamma)-2),\lambda^{\gamma-1}_{2k(\gamma)-2} & & T^\gamma_{k(\gamma)} & \\
& \langle 0, t^\gamma_{k(\gamma)-1}\rangle & & x(2k(1)+\cdots+2k(\gamma)-1),\lambda^{\gamma-1}_{2k(\gamma)-1} & & \langle 1, -\rangle
\end{array}$$

$$\begin{array}{lllll}
x(2k(1)+\cdots+2k(\gamma)),\xi_0 & & x(2k(1)+\cdots+2k(\gamma)+2),\xi_2 & & \ldots. \\
& x(2k(1)+\cdots+2k(\gamma)+1),\xi_1 & & x(2k(1)+\cdots+2k(\gamma)+3),\xi_3 &
\end{array}$$

Let $\vec{Q} = \langle Q_j | \hat{q}_j = 1\rangle$, $\vec{q} = \langle q_j | \hat{q}_j = 0\rangle$, and whenever $\langle \hat{t}^j_{k(j)}, t^j_{k(j)}\rangle = \langle 1, -\rangle$ for $1 \le j \le \delta$,

$$\vec{T_\delta} = \langle T^j_{k(j)} | 1 \le j \le \delta\rangle \text{ and } \vec{t_\delta} = \langle t^j_i | 1 \le j \le \delta \ \&\ 0 \le i < k(j)\rangle.$$

Let $\mu$ be least such that either $\hat{q}_\mu = 0$ or $\mu = \beta + 1$. If $\mu \le \beta$, let $k(0)$ be least such that

$$\mathrm{R}_{C_\mu}(q_\mu, k(0)) \text{ and } \forall j < k(0) \neg R_{C_\mu}(q_\mu, j);$$

otherwise, let $k(0) = m$.

For $\delta \ge 0$, any $\lambda^\delta_i$'s played are properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(\delta)+1)\rangle \text{ using } \langle \omega_{i+1}^{L(\#^1_{\gamma-\delta+1}(\vec{Q}, T^1_{k(1)}, T^2_{k(2)}, \ldots, T^\delta_{k(\delta)}))} | i \le k(\delta)\rangle.$$

Otherwise, the moves of $G^\gamma_{2\beta+1}$ must satisfy all of the conditions required of the moves of $G^\gamma_{2\beta}$: In particular, II must play so that

v.) $$1 \le i < j \le \beta \text{ and } \hat{q}_j = 1 \Rightarrow \hat{q}_i = 1.$$

Furthermore, I must play $Q_j \in L(0^{(\mu-1)\#^1_{\gamma+1}})[\#^1_\gamma]$ whenever $j < \mu$, and if player II plays $\hat{q}_j = 0$ for each $j > i$, then player I must play $Q_i \in L(0^{i\#^1_{\gamma+1}})[\#^1_\gamma]$. Each pair $Q_i$ and $\langle \hat{q}_i, q_i\rangle$ of Borel auxiliary moves is determined by the $\Sigma^0_1$ set $\{m \in \omega | \exists n R_{C_i}(m, n)\}$. If $\hat{t}^j_{k(j)} = 1$ for $j < \delta$, then $G^\gamma_{2\beta+1}$ contains a sequence

$$\langle (T^\delta_k, \langle \hat{t}^\delta_k, t^i_\delta\rangle) | \forall j < k\ \hat{t}^\delta_j = 0\rangle$$

of Borel auxiliary moves, which is related to the $\Pi^0_1$ set $B_\delta$ via $R_\delta$, and player I may only play $T^\delta_k \in L(\vec{Q}, \vec{T}_{\delta-1})[\#^1_{\gamma+1-\delta}]$. Any $\xi_i$'s played must be properly ordered with respect to

$\langle A_\alpha | \alpha < \omega \cdot (k(\gamma)+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T^1_{k(1)}, T^2_{k(2)}, \ldots, T^\gamma_{k(\gamma)}))} | i \leq k(\gamma)\rangle$.

Player I wins $G^\gamma_{2\beta+1}$ iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. $G^\gamma_{2\beta+1}$ is an open game and therefore we define, for each ordinal $\alpha$, $P_\alpha$ as the set of positions with ordinal $\alpha$ and let $P = \bigcup_{\alpha \in \mathrm{ON}} P_\alpha$. If $p$ is a legal position in $G^\gamma_{2\beta+1}$, let $\ell_p$ denote the set of legal positions in $G^\gamma_{2\beta+1}$ consistent with $p$. The set $\ell$ of legal positions for $G^\gamma_{2\beta+1}$ is in $L(0^{(\beta+1)\#^1_{\gamma+1}})$.

Using $\langle P_\alpha | \alpha \in ON\rangle$ and Lemma 0.14, define a wellordering $\prec$ of $\ell$ and the canonical w.s. $s^\gamma_{2\beta+1}$ for $G^\gamma_{2\beta+1}$ so that Lemma 2.5.1 (below) and the following hold: $s^\gamma_{2\beta+1}$ is in $L(0^{(\beta+1)\#^1_{\gamma+1}})$, and if $p$ is a legal position in $G^\gamma_{2\beta+1}$, then $s^\gamma_{2\beta+1}|\ell_p$ is a w.s. for $(G^\gamma_{2\beta+1})_p$ and is definable in any inner model of ZF in which $\prec|\ell_p$ is definable. Analogous to Lemma 2.4.1, $s^\gamma_{2\beta+1}$ has the following properties:

**Lemma 2.5.1.** Let $p$ be a legal position in $G^\gamma_{2\beta+1}$. Recall $\#^1_0(U) = U$ for any $U$. Then $s^\gamma_{2\beta+1}|\ell_p$ is a w.s. for $(G^\gamma_{2\beta+1})_p$ and has the following properties:

vi.) If $1 \leq \tilde{\mu} \leq \beta$, position $p$ includes the moves $\langle \hat{q}_i, q_i\rangle = \langle 0, q_i\rangle$ for $\tilde{\mu} \leq i \leq \beta$, and $n(\tilde{\mu})$ is least such that $R_{C_{\tilde{\mu}}}(q_{\tilde{\mu}}, n(\tilde{\mu}))$, then $s^\gamma_{2\beta+1}|\ell_p$ is definable in $L((\tilde{\mu}-1)\#^1_{\gamma+1}(0))[\#^1_\gamma]$ from $\langle \omega_{i+1}^{L(\tilde{\mu}\#^1_{\gamma+1}(0))} | i \leq n(\tilde{\mu})\rangle$. In particular, if $p$ includes the move $\langle 0, q_\beta\rangle$ and $n$ is least such that $R_{C_\beta}(q_\beta, n)$, then $s^\gamma_{2\beta+1}|\ell_p$ is definable in $L((\beta-1)\#^1_{\gamma+1}(0))[\#^1_\gamma]$ from $\langle \omega_{i+1}^{L(\beta\#^1_{\gamma+1}(0))} | i \leq n\rangle$.

vii.) If $\delta \geq 0$ and $p$ includes the moves $\langle \hat{t}^j_{k(j)}, t^j_{k(j)}\rangle = \langle 1, -\rangle$ for $1 \leq j \leq \delta$, then $s^{\gamma}_{2\beta+1}|\ell_p$ is definable in $L(\vec{Q},\vec{T}_\delta)[\#^1_{\gamma-\delta}]$ from $\langle \omega_{i+1}^{L(\#^1_{\gamma+1-\delta}(\vec{Q},\vec{T}_\delta))}|i \leq k(\delta)\rangle$. In particular, if $p$ includes $\langle \hat{q}_i, q_i\rangle = \langle 1, -\rangle$ for $1 \leq i \leq \beta$, then $s^{\gamma}_{2\beta+1}|\ell_p$ is definable in $L(\vec{Q})[\#^1_\gamma]$ from $\langle \omega^{L(\#^1_{\gamma+1}(\vec{Q}))}|i \leq m\rangle$.

Analogous to the proofs of 2.4.2, by Lemma 2.5.1 and Lemma 2.3.2, we have the following:

**Lemma 2.5.2.** Let $q = (Q_\beta; \langle \hat{q}_\beta, q_\beta\rangle; Q_{\beta-1}; \langle \hat{q}_{\beta-1}, q_{\beta-1}\rangle; ...; Q_1; \langle \hat{q}_1, q_1\rangle)$ be a legal position of $G^{\gamma}_{2\beta+1}$. Let $0 \leq \delta \leq \gamma$ and $p_\delta$ be a legal position of $G^{\gamma}_{2\beta+1}$ which extends $q$ and whose last move is $\langle \hat{t}^{\delta}_{k(\delta)}, t^{\delta}_{k(\delta)}\rangle = \langle 1, -\rangle$. If

$$A' = A(\vec{Q}, \vec{T}_\delta; \vec{q}, \vec{t}_\delta, \bar{x}(2k_1 + \cdots + 2k_\delta)),$$

then the following hold:

viii.) The w.s. $s^{\gamma}_{2\beta+1}|\ell_{p_\delta}$ can be integrated so as to obtain a w.s.

$$s' \in L(\#^1_{\gamma+1-\delta}(\vec{Q},\vec{T}_\delta))$$

for $A'$ and $s'$ is a w.s. of the player for whom $s^{\gamma}_{2\beta+1}$ is a w.s. In particular, for $\delta = 0$, we have that the w.s. $s^{\gamma}_{2\beta+1}|\ell_q$ can be integrated so as to obtain a w.s. $s' \in L(\#^1_{\gamma+1}(\vec{Q}))$ for $A(\vec{Q};\vec{q})$ and $s'$ is a w.s. of the player for whom $s^{\gamma}_{2\beta+1}$ is a w.s.

ix.) If $s^{\gamma}_{2\beta+1}$ is a w.s. for I, $p$ is a position consistent with $s'$, and the moves in $p$ of player II are consistent with $\vec{q}$, $\vec{t}_\delta$, and $\bar{x}(2k_1 + 2k_2 + \cdots + 2k_\delta)$, then

$$p \in Q_i \text{ for } i \text{ such that } \hat{q}_i = 1 \text{ and}$$

$$p \in T^{j}_{k(j)} \text{ for } 1 \leq j \leq \delta.$$

Therefore, if $s^{\gamma}_{2\beta+1}$ is a w.s. for I and $x$ is a play consistent with $s'$, then

$x \in A(\vec{q}, \vec{t}_\delta, \bar{x}(2k_1 + \cdots + 2k_\delta))$.

x.) Let $p = (x(0); x(1); x(2); ...; x(i))$ be a position consistent with $s'$ and with both $\vec{Q}$ and $\vec{T}_\delta$. If the moves in $p$ of the player for whom $s'$ is not a w.s. are consistent with $\vec{q}$, $\vec{t}_\delta$, and $\bar{x}(2k_1 + 2k_2 + \cdots + 2k_\delta)$, then $p$ is consistent with $\vec{q}$, $\vec{t}_\delta$, and $\bar{x}(2k_1 + 2k_2 + \cdots + 2k_\delta)$.

By Induction Hypothesis, we have the following:

**Lemma 2.5.3.** If $q = (Q_\beta; \langle 0, q_\beta \rangle)$ is a legal position of $G^\gamma_{2\beta+1}$, then $A(q_\beta)$ has a w.s. $s' \in L(\beta\#^1_{\gamma+1}(0))$ such that the following hold:

(1) $s'$ is a w.s. of the player for whom $s^\gamma_{2\beta+1}$ is a w.s.,

(2) if $p$ is a position of $G_A$ such that $p$ is consistent with $s'$ and the moves in $p$ of player II are consistent with $q_\beta$, then $p$ is consistent with $q_\beta$.

**Claim I:** Player I has a w.s. for $G_A$ if he has one for $G^\gamma_{2\beta+1}$.

Let's first consider the case in which $\langle\, \rangle \in P$. Then $s^\gamma_{2\beta+1} \in L(0^{\beta\#^1_{\gamma+1}})[\#^1_\gamma]$ is a w.s. for I in $G^\gamma_{2\beta+1}$. We use $s^\gamma_{2\beta+1}$ to define a w.s. $s$ for I in $G_A$. For $1 \leq i \leq \beta$, let

$$Q_i = s^\gamma_{2\beta+1}(Q_\beta; \langle 1, - \rangle; Q_{\beta-1}; \langle 1, - \rangle; ...; Q_{i+1}; \langle 1, - \rangle), \ \ \langle \hat{q}_i, q_i \rangle = \langle 1, - \rangle,$$
$$\text{and } p_0 = (Q_\beta; \langle 1, - \rangle; Q_{\beta-1}; \langle 1, - \rangle; ...; Q_1; \langle 1, - \rangle).$$

Then $\vec{Q} = \langle Q_i | 1 \leq i \leq \beta \rangle$. By Lemma 2.5.1(vii), $s^\gamma_{2\beta+1} | \ell_{p_0}$ is a w.s. for $(G^\gamma_{2\beta+1})_{p_0}$ and is definable in $L(\vec{Q})[\#^1_\gamma]$ from $\langle \omega_{i+1}^{L(\#^1_{\gamma+1}(\vec{Q}))} | i \leq m \rangle$. By Lemma 2.5.2(viii), obtain a w.s. $s_0 \in L(\#^1_{\gamma+1}(\vec{Q}))$ for $A(\vec{Q})$ by integrating the w.s. $s^\gamma_{2\beta+1} | \ell_{p_0}$ for $A(\vec{Q})$, and let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); ...; x(i-$

1)) such that $\forall i' \leq i \forall n \neg R_{C_j}(\bar{x}(i'), n)$ for $1 \leq j \leq \beta$.

Suppose we reach a position such that $\exists i \exists j \exists n R_{C_j}(\bar{x}(i), n)$. Since $C_j \subseteq C_\beta$, there is a least $i$ such that $\exists n R_{C_\beta}(\bar{x}(i), n)$. By (iv), $i$ is odd. Let $Q_\beta = s(\ )$ (as above), $\langle \hat{q}_\beta, q_\beta \rangle = \langle 0, \bar{x}(i) \rangle$, and $q = (Q_\beta; \langle \hat{q}_\beta, q_\beta \rangle)$. By Lemma 2.5.3, player I has a w.s. $s_1 \in L(\beta \#^1_{\gamma+1}(0))$ for $A(q_\beta)$. Let $s(p) = s_1(p)$ for any position $p$ which extends $(x(0); x(1); x(2); ...; x(i-1))$.

**Claim:** The strategy $s$ of player I is a w.s. in $G_A$.

Suppose $x$ is a play of $G_A$ consistent with $s$. By the definition of $s$, there exists a w.s. $s_0 \in L(\#^1_{\gamma+1}(\vec{Q}))$ for $A(\vec{Q})$ and $x(2i) = s_0(x(1); x(3); ...; x(2i-1))$ whenever $\forall i' \leq 2i \forall n \neg R_{C_j}(\bar{x}(i'), n)$ for $1 \leq j \leq \beta$. If $\forall i \forall n \neg R_{C_j}(\bar{x}(i), n)$ holds (for $1 \leq j \leq \beta$), then $x \in A(\vec{Q})$ so that $x \in A$ by Lemma 2.5.2(ix).

Otherwise, $\exists n R_{C_\beta}(\bar{x}(i), n)$ for some least $i$. By the definition of $s$, there exists a w.s. $s_1 \in L(\beta \#^1_{\gamma+1}(0))$ for $A(\bar{x}(i))$ such that

$$x(2k) = s_1(x(1); x(3); ...; x(2k-1)) \text{ whenever } 2k > i.$$

Therefore, $x \in A(\bar{x}(i))$ so that $x \in A$. Thus, $x$ is a win for I.

**Claim II:** Player II has a w.s. for $G_A$ if he has one for $G^\gamma_{2\beta+1}$.

Now let's consider the case $\langle\ \rangle \notin P$. We integrate II's w.s. $s^\gamma_{2\beta+1}$ for $G^\gamma_{2\beta+1}$ to get the w.s. $s \in L(0^{(\beta+1)\#^1_{\gamma+1}})$ for II in $G_A$. Let

$$Q_\beta = \{\text{positions } q \text{ in } G_A | \forall Q' \in L(\beta \#^1_{\gamma+1}(0))[\#^1_\gamma]\ \langle 0, q \rangle \neq s^\gamma_{2\beta+1}(Q')\}$$

and $Q_i = Q_\beta$ for $1 \leq i < \beta$. Clearly $\langle 1, - \rangle = s^\gamma_{2\beta+1}(Q_\beta)$. Furthermore, for $1 \leq i \leq \beta$,

$Q_i \in L(0^{\beta\#^1_{\gamma+1}})[\#^1_\gamma]$, and by (v), $\langle 1, -\rangle = s^\gamma_{2\beta+1}(Q_\beta; Q_{\beta-1}; Q_{\beta-2}; ...; Q_i)$. Let $p_0 = (Q_\beta; \langle 1, -\rangle; Q_{\beta-1}; \langle 1, -\rangle; Q_{\beta-2}; \langle 1, -\rangle; ...; Q_1; \langle 1, -\rangle)$. By Lemma 2.5.2(viii), integrate $s^\gamma_{2\beta+1}|\ell_{p_0}$ so as to obtain a w.s. $s_0 \in L(\#^1_{\gamma+1}(\vec{Q}))$ for the game $A(\vec{Q})$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that $\forall i' \leq i\ \bar{x}(i') \in Q_\beta = \bigcap_{1\leq j\leq\beta} Q_j$.

Suppose we reach a position $\bar{x}(i)$ (of least length) such that $\bar{x}(i) \notin Q_\beta$. Then $i$ is odd and there exists $Q'_\beta \in L(0^{\beta\#^1_{\gamma+1}})[\#^1_\gamma]$ such that the position $p_1 = (Q'_\beta; \langle 0, \bar{x}(i)\rangle)$ is consistent with $s^\gamma_{2\beta+1}$. By Lemma 2.5.3, player II has a w.s. $s_1 \in L(\beta\#^1_{\gamma+1}(0))$ for $A(\bar{x}(i))$. Let $s(p) = s_1(p)$ for any position $p$ which extends $\bar{x}(i)$.

**Claim:** The strategy $s$ of player II is a w.s. for $G_A$.

Suppose $x$ is a play of $G_A$ consistent with $s$. By the definition of $s$, there exists a w.s. $s_0 \in L(\#^1_{\gamma+1}(\vec{Q}))$ for $A(\vec{Q})$ and $x(2i+1) = s_0(x(0); x(2); ...; x(2i))$ whenever $\forall i' \leq 2i+1\ \bar{x}(i') \in Q_\beta = \bigcap_{1\leq j\leq\beta} Q_j$. If $\forall i\ \bar{x}(i) \in Q_\beta$, then $x \notin A(\vec{Q})$ so that $x \notin A$.

Otherwise, there exists (odd) $i$ such that $\bar{x}(2i) \notin Q_\beta$ and $\forall j < i\ \bar{x}(j) \in Q_\beta$. By the definition of $s$, there exists a w.s. $s_1 \in L(\beta\#^1_{\gamma+1}(0))$ for $A(\bar{x}(i))$ such that $x(2k+1) = s_1(x(0); x(2); ...; x(2k))$ whenever $2k+1 \geq i$. Therefore, since $s_1$ is a w.s. for II, $x \notin A(\bar{x}(i))$ so that $x \notin A$. Consequently, $s$ is a w.s. in $G_A$ of the player for whom $s^\gamma_{2\beta+1}$ is a w.s. ■

**Definition 2.5.** Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi^0_1$, $C_1, C_2, C_3, ..., C_\beta \in \Sigma^0_1$,

$\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ strongly witness $A \in (\gamma * \Pi^0_1, \beta * \Sigma^0_1)^*_+$. Then we refer to the auxiliary game $G^\gamma_{2\beta+1}$ described in the Proof of Theorem 2.5 as *the* $G^\gamma_{2\beta+1}$ *auxiliary game determined by*

$$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi^0_1,\ C_1, C_2, C_3, ..., C_\beta \in \Sigma^0_1,\ \langle A_\alpha \mid \alpha < \omega^2\rangle, \textit{ and } m \in \omega.$$

Suppose $\vec{U} = \langle U_i | i < \beta\rangle$ and $\vec{u} = \langle u_i | i < \gamma\rangle$ respectively are a finite sequence of I-imposed subgames of $G_A$ and a sequence of legal positions of $G_A$. Then *the* $G^\gamma_{2\beta+1}(\vec{U};\vec{u})$ *auxiliary game determined by*

$$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1,\ C_1, C_2, C_3, ..., C_\beta,\ \langle A_\alpha \mid \alpha < \omega^2\rangle, \textit{ and } m \in \omega$$

is the game in which player I wins iff a position is reached at which II cannot make a (legal) move, which has exactly the same moves as $G^\gamma_{2\beta+1}$, and these moves are subject to the following conditions:

i.) $\{q | \exists k R_{C_i}(q, k)\}$ determines the Borel auxiliary moves $Q_i$ and $\langle \hat{q}_i, q_i\rangle$ for $1 \le i \le \beta$.

ii.) The sequence $\langle (T^\delta_n; \langle \hat{t}^\delta_n, t^\delta_n\rangle) | \forall j < n\ \hat{t}^\delta_j = 0\rangle$ of Borel auxiliary moves and the $\Pi^0_1$ set $B_\delta$ are related via $R_{B_\delta}$.

iii.) If $1 \le i < j \le \beta$ and $\hat{q}_j = 1$, then $\hat{q}_i = 1$.

iv.) Each $\bar{x}(i) \in \bigcap_{j<\beta} U_j$ and each $\bar{x}(i)$ must be consistent with every $u_j$.

v.) Let $\vec{Q} = \langle Q_i | \hat{q}_i = 1\rangle$, $\vec{q} = \langle q_i | \hat{q}_i = 0\rangle$, and $\mu$ is least such that $\mu = \beta + 1$ or $\hat{q}_\mu = 0$. Then

$Q_i \in L(i\#^1_{\gamma+1}(\vec{U}))[\#^1_\gamma]$ for $i \ge \mu$ and $Q_i \in L((\mu - 1)\#^1_{\gamma+1}(\vec{U}))[\#^1_\gamma]$ for $i < \mu$.

vi.) $T^\delta_i \in L(\vec{U}, \vec{Q}; \vec{T}_{\delta-1})[\#^1_{\gamma+1-\delta}]$.

vii.) If $\hat{q}_\beta = 0$, let $k(0)$ be least such that $R_{C_\mu}(q_\mu, k(0))$; otherwise, let $k(0) = m$. The $\lambda_i^\delta$'s are properly ordered with repect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(\delta)+1)\rangle \text{ using } \langle \omega_{i+1}^{L(\#^1_{\gamma+1-\delta}(\vec{U},\vec{Q},\vec{T}_\delta))} | i \leq k(\delta)\rangle$$

viii.) If $\hat{t}^\gamma_{k(\gamma)} = 1$, the $\xi_i$'s are properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (k(\gamma)+1)\rangle \text{ using } \langle \omega_{i+1}^{L(\#^1_1(\vec{U},\vec{Q},\vec{T}))} | i \leq k(\gamma)\rangle.$$

These conditions are analogous to the conditions for the moves of $G^\gamma_{2\beta+1}$. The first three are conditions which the moves of $G^\gamma_{2\beta+1}$ also must satisfy. The others are derived by changing the conditions for the moves of $G^\gamma_{2\beta+1}$ so that we obtain conditions which are consistent with $\vec{U}$ and $\vec{u}$. We refer to $G^\gamma_{2\beta+1}(\vec{U};\vec{u})$ instead of the $G^\gamma_{2\beta+1}(\vec{U};\vec{u})$ auxiliary game determined by $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1,\ C_1, C_2, C_3, ..., C_\beta,\ \langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ whenever $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1,\ C_1, C_2, C_3, ..., C_\beta,\ \langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ are clear from the context.

Analogous to Theorems 2.4 and 2.5, we have Corollaries 2.4.1 and 2.5.1 (below). The proof of these two corollaries is similar to the (inductive) proof of Theorems 2.4 and 2.5. However, we instead use the following stronger induction hypothesis:

**Induction Hypothesis for Corollaries 2.4.1 and 2.5.1.** Suppose $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1 \in \Pi^0_1,\ C_1, C_2, C_3, ..., C_{\beta-1} \in \Sigma^0_1,\ \langle A_\alpha | \alpha < \omega^2\rangle$, and $m \in \omega$ witness that $A$ is $(\gamma * \Pi^0_1, (\beta-1) * \Sigma^0_1)^*_+$. Also, suppose $\vec{U}$ is a finite sequence of I-imposed subgames of $G_A$, $\vec{u}$ is a sequence of legal positions in $G_A$ of odd length, $\vec{U}$ has a definable wellordering in $L(\vec{U})$, and $\beta\#^1_{\gamma+1}(\vec{U})$ exists. Let $s^\gamma_{2\beta-1}$ be the $G^\gamma_{2\beta-1}$ auxiliary game determined by

$$B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1,\ C_1, C_2, C_3, ..., C_{\beta-1},\ \langle A_\alpha | \alpha < \omega^2\rangle, \text{ and } m.$$

Then $s^\gamma_{2\beta-1}$ can be integrated so as to obtain a w.s. $s \in L(\beta\#^1_{\gamma+1}(\vec{U}))$ for $A(\vec{U};\vec{u})$ such that the following hold:

i.) $s$ is a w.s. of the player for whom $s^\gamma_{2\beta-1}$ is a w.s.

ii.) If $s$ is a w.s. for I, $p$ is a position consistent with $s$, and the moves in $p$ of player II are consistent with $\vec{u}$, then $p \in \vec{U}$. Therefore, if $x$ is a w.s. for I and $x$ is a position consistent with $s$, then $x \in A(\vec{u})$.

iii.) If $p$ is a (legal) position of the game $A(\vec{U})$ such that the moves in $p$ of the player for whom $s$ is not a w.s. are consistent with $\vec{u}$, then $p$ is consistent with $\vec{u}$.

Analogous to Theorem 2.5, we have the following:

**Corollary 2.5.1.** Let $B_\gamma, B_{\gamma-1}, B_{\gamma-2}, ..., B_1,\ C_1, C_2, C_3, ..., C_\beta,\ \langle A_\alpha \mid \alpha < \omega^2\rangle$, $m$, $A$, $\vec{U}$, and $\vec{u}$ be as in Definition 2.5. Let $p$ be a legal position of a game $G^*$ such that the moves of $G^*$ following $p$ constitute a play of $G^\gamma_{2\beta+1}(\vec{U};\vec{u})$. Suppose $\vec{U}$ has a wellordering which is definable in $L(\vec{U})$, $(\beta+1)\#^1_\gamma(\vec{U})$ exists, and $s^*$ is a w.s. for $G^*$ such that $s^*|\ell_p \in L(\beta\#^1_{\gamma+1}(\vec{U}))[\#^1_\gamma]$. Then $s^*|\ell_p$ can be integrated so as to obtain a w.s. $s_p \in L((\beta+1)\#^1_{\gamma+1}(\vec{U}))$ for $A(\vec{U};\vec{u})$ such that the following hold:

i.) $s_p$ is a w.s. of the player for whom $s^*$ is a w.s.

ii.) If $s^*$ is a w.s. for I, $\hat{p}$ is a position consistent with $s_p$, and the moves in $\hat{p}$ of player II are consistent with $\vec{u}$, then $\hat{p} \in \bigcap_{i<\beta} U_i$. Therefore, if $s^*$ is a w.s. for I and $x$ is a play consistent with $s_p$, then $x \in A(\vec{u})$.

iii.) Let $\hat{p}$ be a position consistent with $s_p$ and with $\vec{U}$. If the moves in $\hat{p}$ of the player for whom $s_p$ is not a w.s. are consistent with $\vec{u}$, then $\hat{p}$ is consistent with $\vec{u}$. ∎

By the comment preceding Section 0.3, we have the following corollary to Theorems 2.4 and 2.5:

**Corollary 2.6.** Let $\langle \Gamma_i | 1 \leq i \leq k \rangle$ be a sequence such that the following holds: there exist $i_1 < i_2 < i_3 < \cdots < i_\gamma$ such that $k = i_\gamma + \beta$, $\Gamma_i = \Pi^0_1$ for $i = i_1, i_2, i_3, ..., i_\gamma$, and otherwise $\Gamma_i = \Sigma^0_1$.

i.) If $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma] \models$ "$r^{\#^1_{\gamma+1}}$ exists for every real $r$," then every $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)^*$ game has a w.s. in $L(\beta\#^1_{\gamma+1}(0))[\#^1_\gamma]$.

ii.) If $(\beta+1)\#^1_{\gamma+1}(0)$ exists, then every $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)^*_+$ game has a w.s. in $L((\beta+1)\#^1_{\gamma+1}(0))$. ∎

In [Du4], we prove the converse of Theorem 2.5 and Corollary 2.5.1. Therefore, the converse of all Theorems $x.y$ (of this paper), in which $x \neq 0$ and $y$ is odd, are true. In subsequent papers, we generalize the results of this paper as well as [Du 1,2,3,4]. In this paper, we characterized the determinacy of $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_n)^*$ games as well as the determinacy of $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_n)^*_+$ games for $\{\Gamma_i | i \leq n\} \subseteq \{\Pi^0_1, \Sigma^0_1\}$. We characterized the determinacy of these games in terms of least inner models of "ZFC + $F(r)$ exists for every real $r$," where $F$ is some extended sharp function on the reals. In subsequent papers (e.g. [Du 5]), we characterize the determinacy of $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)^*$ games as well as the determinacy of $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)^*_+$ games for $\{\Gamma_i | i \leq k\} \subseteq \{\Pi^0_n, \Sigma^0_n | n \in \mathbf{N}\}$. These characterizations are natural generalizations of the results in this paper. If $\{\Gamma_i | i \leq k\} \subseteq \{\Pi^0_n, \Sigma^0_n | n \leq m\}$ and $A$ is either $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)^*$ or $(\Gamma_1, \Gamma_2, \Gamma_3, ..., \Gamma_k)^*_+$, we relate the determinacy of the game $G_A$ to the existence of some least inner model of "$F(r)$ exists for every object $r$ of type $m$," where $F$ is some extended sharp function on objects of type $m$.

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