Since we are only interested in the classes $\left(\vec{\Gamma}\right)^*$ and $\left(\vec{\Gamma}\right)^*_+$ for which the individual $\Gamma_i \in \left\{\Pi^0_\alpha, \Sigma^0_\alpha, \Pi^1_1 | \alpha < \omega_1^{CK}\right\}$, it is enough for us to consider $\left(\vec{\Gamma}\right)$ of the form

$(**)$ $\left(\gamma(0) * \Pi^1_1, \gamma(1) * \Sigma^0_{k(1)}, \gamma(2) * \Sigma^0_{k(2)}, \cdots, \gamma(\ell) * \Sigma^0_{k(\ell)}\right)$

where $\gamma(0), \gamma(1), \cdots, \gamma(\ell) < \omega_1^{CK}$ and $\omega_1^{CK} > k(1) > k(2) > \cdots > k(\ell)$.

This is because:

(i) $(\cdots, \Gamma_1, \Gamma_2, \cdots) = (\cdots, \Gamma_2, \cdots)$ if $\Gamma_1 \subseteq \Delta(\Gamma_2)$ and Γ_2 has sufficient closure properties.

(ii) $\left(\Pi^0_\alpha\right) = \left(\Sigma^0_{\alpha+1}\right)^*$ and $\left(\Pi^0_\alpha\right)^*_+ = \left(\Sigma^0_{\alpha+1}\right)^*_+$.

In the interest of minimizing the Γ_i in $\left(\vec{\Gamma}\right)$, it is customary to consider $\left(\vec{\Gamma}\right)$ of the form:

$(***)$ $\left(\gamma(0) * \Pi^1_1, \gamma(1) * \Gamma_{k(1)}, \gamma(2) * \Gamma_{k(2)}, \cdots, \gamma(\ell) * \Gamma_{k(\ell)}\right)$

where $\gamma(0), \gamma(1), \gamma(2), \cdots, \gamma(\ell) < \omega_1^{CK}$, $\omega_1^{CK} > k(1) > k(2) > k(3) > \cdots > k(\ell)$ and

$$\Gamma_{k(i)} = \begin{cases} \Pi^0_{k(i)} & \text{if} \quad \alpha(i) < \omega \\ \Pi^0_{k(i)-1} & \text{if} \quad \alpha(i) \text{ is an infinite successor ordinal} \\ \Sigma^0_{k(i)} & \text{otherwise.} \end{cases}$$

By (ii), $(**)$ and $(***)$ result in the same $\left(\vec{\Gamma}\right)^*$ and $\left(\vec{\Gamma}\right)^*_+$.

We actually prove (from the appropriate large cardinal property) the determinacy of the relativized analogues of $\left(\vec{\Gamma}\right)^*$ and $\left(\vec{\Gamma}\right)^*_+$ for $\vec{\Gamma}$ as in $(***)$. This simplifies the proofs of Sections 2 and 3 by providing a stronger (relativized) induction hypotheses.

Definition 1.18. $\left(\vec{\Gamma}\right)^*(z)$, $\left(\vec{\Gamma}\right)^*_+(z)$. Suppose $\beta < \omega_1^{CK}(z)$ and $\forall i < \beta$

$$\Gamma_i(z) \in \left\{\Pi^0_\alpha(z), \Sigma^0_\alpha(z), \Pi^1_{n+1}(z), \Sigma^1_{n+1}(z) | \alpha < \omega_1^{CK}(z), n < \omega\right\}.$$

Define $(\langle\Gamma_i(z) | i < \beta\rangle)(z)$ to be the collection of all $(\langle B_i | i < \beta\rangle)$ for which $\langle B_i | i < \beta\rangle$ is a $\Pi^1_1(z)$ sequence witnessing $(\langle B_i | i < \beta\rangle) \in (\langle\Gamma_i(z) | i < \beta\rangle)$.

Let $(\langle\Gamma_i(z) | i < \beta\rangle)^*(z)$ [respectively $(\langle\Gamma_i(z) | i < \beta\rangle)^*_+(z)$] be the collection of $B^*\left(\vec{A}\right)$ [respectively $B^*\left(\vec{A}, D\right)$] for which $B \in (\langle\Gamma_i(z) | i < \beta\rangle)(z)$ and $\vec{A}$ witnesses that some set is $\omega^2 - \Pi^1_1(z)$ [respectively, and $D \in < \omega^2 - \Pi^1_1(z)$]. △

As expected, we speak of "witnessing" with respect to the relativized classes of Definition 1.18.

Notation. We abbreviate $(\langle\Gamma_i(z) | i < \beta\rangle)(z)$, $(\langle\Gamma_i(z) | i < \beta\rangle)^*(z)$, and $(\langle\Gamma_i(z) | i < \beta\rangle)^*_+(z)$ by $(\langle\Gamma_i | i < \beta\rangle)(z)$, $(\langle\Gamma_i | i < \beta\rangle)^*(z)$ and $(\langle\Gamma_i | i < \beta\rangle)^*_+(z)$, suppressing the "$(z)$" in $\Gamma_i(z)$ in these cases.

In practice this causes no confusion, but our abbreviation leads for instance to our writing Π^0_α even when $\omega_1^{CK} \leq \alpha < \omega_1^{CK}(z)$ (in which case Π^0_α is not defined). We could avoid this by using the more cumbersome notation, or by only defining our relativized classes (and proving theorems about them) only for $\alpha < \omega_1^{CK}$. The reader can read this paper with this last option in mind.

Normal Form Theorem 1.19. Let x be a real. Let $\left(\vec{\Gamma}\right) = \left(\gamma(0) * \Pi^1_1(x), \gamma(1) * \Gamma_{k(1)}, \gamma(2) * \Gamma_{k(2)}, \cdots, \gamma(\ell) * \Gamma_{k(\ell)}\right)$, where $\forall i \leq \ell \; k(i), \gamma(i) < \omega_1^{CK}(x)$ and for $k < \omega_1^{CK}(x)$,

$$\Gamma_k = \begin{cases} \Sigma^0_1(x) & \text{if } k = 0, \\ \Pi^0_k(x) \text{ or } \Sigma^0_{k+1}(x) & \text{if } 0 < k < \omega, \\ \Sigma^0_k(x) & \text{if } k \text{ is an infinite limit ordinal,} \\ \Pi^0_{k-1}(x) \text{ or } \Sigma^0_k(x) & \text{if } k \text{ is an infinite successor ordinal.} \end{cases}$$

If $A \in \left(\vec{\Gamma}\right)^*(x)$, then this is witnessed by some $\vec{B} = \langle B_\alpha | \alpha < \gamma(0) + \gamma(1) + \cdots + \gamma(\ell)\rangle$ and $\vec{A} = \left\langle A_\alpha | \alpha < \omega^2\right\rangle$ which satisfy the following:

(i) $A_\beta \supseteq A_\alpha$ if $\beta < \alpha < \omega^2$.

(ii) $\bigcap_{\alpha<\omega^2} A_\alpha = \emptyset$.

(iii) $\langle B_\alpha | \alpha < \gamma(0)\rangle$ is an increasing sequence (of $\Pi^1_1(x)$ sets).

(iv) $B_\alpha \subseteq B_{\alpha+1}$ whenever $\gamma(0) + \gamma(1) + \gamma(2) + \cdots + \gamma(i) \leq \alpha < \alpha + 1 < \gamma(0) + \gamma(1) + \cdots + \gamma(i+1)$.

If $A \in \left(\vec{\Gamma}\right)^*_+(x)$, then this is witnessed by some $\vec{B}$, $\vec{A}$, and $\vec{D} = \langle D_\alpha | \alpha < \omega \cdot m\rangle$ (for some $m < \omega$) such that (i) – (iv) hold and

(v) each $D_\alpha \supseteq \left\{x | \exists n \left(\vec{B}\right)(x, n)\right\}$. ■

Our Normal Form Theorem remains true if we replace (iii) by

(iii)′ $\forall \alpha < \gamma(0) \; B_\alpha \in \Pi^1_1(x)$, $\forall \alpha < \beta < \gamma(0) \; \exists n B_\alpha(x, n) \Rightarrow \exists n B_\beta(x, n)$, and each B_α is uniformized, i.e. $\forall x$ $\exists$ at most one n such that $B(x, n)$.

By the definition of witness, $B_\alpha \in \Pi^1_1(z)$ for $\alpha < k(0)$ and $B_\alpha \in \Gamma_{k(i+1)}(z)$ for $\alpha \in \Big[\gamma(0) + \gamma(1) + \cdots + \gamma(i), \gamma(0) + \gamma(1) + \cdots + \gamma(i+1)\Big)$.

We would like $\vec{B}$ to be increasing, but cannot hope to include such a requirement in the Normal Form Theorem. In particular, we cannot require $B_{\gamma(0)+\gamma(1)+\cdots+\gamma(i+1)} \supseteq \bigcup_{\alpha<\gamma(0)+\gamma(1)+\cdots+\gamma(i+1)} B_\alpha$ (note that $\Gamma_{\gamma(0)+\gamma(1)+\cdots+\gamma(i+1)}$ strictly contains $\bigcup_{\alpha<\gamma(0)+\gamma(1)+\cdots+\gamma(i+1)} \Gamma_\alpha$). Also whenever $\gamma(i+1) > \omega$, we cannot replace (iv) of our Normal Form Theorem by $\left\langle B_\alpha \middle| \sum_{j\leq i} \gamma(j) \leq \alpha < \sum_{j\leq i+1} \gamma(j)\right\rangle$ is an increasing sequence (note that the union of the components from a $\Pi^1_1(z)$ sequence of $\Pi^0_k(z)$ sets may not be $\Pi^0_k(z)$).

Let $\left(\vec{\Gamma}\right) = \left(\vec{\Gamma}_H, \gamma(\ell-1) * \Gamma_{k(\ell-1)}, \gamma(\ell) * \Gamma_{k(\ell)}\right) = \left(\vec{\Gamma}_H, \gamma(\ell-1) * \Pi^0_1, \gamma(\ell) * \Sigma^0_1\right)$ be as in the Normal Form Theorem with $k(\ell-1) = 1$ and $k(\ell) = 0$. In this paper, we prove $\left(\vec{\Gamma}\right)^*(r)$ and $\left(\vec{\Gamma}\right)^*_+(r)$ determinacy each follow from the corresponding large cardinal property, under the hypothesis that for each $\hat{r}$, $\left(\vec{\Gamma}_H\right)^*_+(\hat{r})$ determinacy follows from the existence of indiscernibles for the correct model. With this as our goal, we shall only explicitly cite (from the Normal Form Theorem) (v) and that (iv) holds for the $\Pi^0_1(z)$ and $\Sigma^0_1(z)$ sets, i.e. (iv) holds when $i = \ell - 2, \ell - 1$.

The Normal Form Theorem with (iii)$'$ in place of (iii) is what we use in our proof of this paper's hypothesis and therefore (implicitly) in our proof of $\left(\vec{\Gamma}\right)^{*}(r)$ and $\left(\vec{\Gamma}\right)^{*}_{+}(r)$ determinacy. More specifically, in [Du7], we inductively prove $\left(k(0) * \Pi^1_1\right)^{*}_{+}(r)$ determinacy (from $\#^{(k(0)+1)\infty}(r)$ exists), and only cite the uniformization of the last $\Pi^1_1(z)$ set $B_{\gamma(0)-1}$ when $\gamma(0)$ is a successor. This uniformization is needed to invoke quasi-Borel determinacy; the induction hypothesis takes care of the $\Pi^1_1(z)$ sets B_α for $\alpha + 1 < k(0)$.

Possibly see page 17 of 4aa.tex for strongly witness and Normal Form Theorem.

Our original interest in writing this paper was to show the following two results for $r = \emptyset$:

(i) If $L\left(\beta\#^1_{\gamma+1}(r)\right)\left[\vec{\#}^1_{\gamma}\right] \models$ "$\forall i < \gamma\ \#^1_{i+1}$ is total", then every $\left(\gamma * \Pi^0_1, \beta * \Sigma^0_1\right)^{*}(r)$ game is determined and has a w.s. in $L\left((\beta+1)\#^1_{\gamma+1}(r)\right)$.

(ii) $(\beta+1)\#^1_{\gamma+1}(r)$ exists (i.e. $L\left(\beta\#^1_{\gamma+1}(r)\right)\left[\vec{\#}^1_{\gamma}\right]$ has indiscernibles) iff every $\left(\gamma * \Pi^0_1, \beta * \Sigma^0_1\right)^{*}_{+}(r)$ game is determined, in which case each such game has a w.s. in $L\left((\beta+1)\#^1_{\gamma+1}(r)\right)$.

Sections 2 and 3 are written so that the reader may in fact concentrate exclusively on proving (i) and (ii). However, the proof contributes to showing a more general result, Theorem 1.21 below.

Definition 1.20. Set $\Gamma_r\left(\#^{\infty}\right) =_{df} \Pi^1_1(r)$, and for $k < \omega_1^{CK}(r)$, fix

$$\Gamma_r\left(\#^k\right) =_{df} \begin{cases} \Pi^0_k(r) \ \text{ or } \ \Sigma^0_{k+1}(r) & \text{if } k < \omega \\ \Sigma^0_k(r) & \text{if } k \text{ is a limit ordinal} \\ \Pi^0_{k-1}(r) \ \text{ or } \ \Sigma^0_k(r) & \text{if } k \text{ is an infinite successor ordinal.} \end{cases}$$

For $0 \le \gamma(i) < \omega_1^{CK}(r)$ and $\omega_1^{CK}(r) > k(1) > k(2) > \cdots > k(n)$, set

$$\Gamma_r\left(\vec{\mathcal{S}}^{\gamma(n)}_{k(n)}\vec{\mathcal{S}}^{\gamma(n-1)}_{k(n-1)}\cdots\vec{\mathcal{S}}^{\gamma(1)}_{k(1)}\vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset)\right) =_{df} \left(\gamma(0) * \Gamma_r\left(\#^{\infty}\right), \gamma(1) * \Gamma_r\left(\#^{k(1)}\right), \gamma(2) * \Gamma_r\left(\#^{k(2)}\right), \cdots, \gamma(n) * \Gamma_r\left(\#^{k(n)}\right)\right).$$

△

Theorem 1.21. Let r be a real and $\beta < \omega_1^{CK}(r)$. Let $\vec{\#} = \vec{\mathcal{S}}^{\gamma(n)}_{k(n)}\vec{\mathcal{S}}^{\gamma(n-1)}_{k(n-1)}\cdots\vec{\mathcal{S}}^{\gamma(1)}_{k(1)}\vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset) \neq \emptyset$, where $0 \le \gamma(i) < \omega_1^{CK}(r)$ and $\omega_1^{CK}(r) > k(1) > k(2) > \cdots > k(n)$.

(iii) If one of $\gamma(n), \gamma(n-1), \cdots, \gamma(1)$ is nonzero (i.e. if $\vec{\#} \neq \#^{\gamma(0)\infty}$) and $L\left(\left(\beta\mathcal{S}_1\left(\vec{\#}\right)\right)(r)\right)\left[\vec{\#}\right] \models$ " each $\#$ in $\vec{\#}$ is total", then every $\left(\Gamma_r\left(\vec{\#}\right), \beta * \Sigma^0_1(r)\right)^{*}(r)$ is determined and has a w.s. in $L\left(\left(\beta\mathcal{S}_1\left(\vec{\#}\right)\right)(r)\right)\left[\vec{\#}\right]$.

(iv) $\left((\beta+1)\mathcal{S}_1\left(\vec{\#}\right)\right)(r)$ exists (i.e. $L\left(\left(\beta\mathcal{S}_1\left(\vec{\#}\right)\right)(r)\right)\left[\vec{\#}\right]$ has indiscernibles) iff every $\left(\Gamma_r\left(\vec{\#}\right), \beta * \Sigma^0_1(r)\right)^{*}_{+}(r)$ game is determined, in which case each such game has a w.s. in $L\left(\left((\beta+1)\mathcal{S}_1\left(\vec{\#}\right)\right)(r)\right)$.

In particular:

Corollary 1.21.1. Let r and $\vec{\#}$ be as in Theorem 1.21.

(v) If one of $\gamma(n), \gamma(n-1), ..., \gamma(1)$ is nonzero (i.e. if $\vec{\#} \neq \#^{\gamma(0)\infty}$) and $L(r)\left[\vec{\#}\right] \models$ " each $\#$ in $\vec{\#}$ is total", then every $\left(\Gamma_r\left(\vec{\#}\right)\right)^{*}(r)$ is determined and has a w.s. in $L(r)\left[\vec{\#}\right]$.

(vi) If $L(r)\left[\vec{\#}\right]$ has indiscernibles, then each $\left(\Gamma_r\left(\vec{\#}\right)\right)^{*}_{+}(r)$ game has a w.s. in $L\left(\left(\mathcal{S}_1\left(\vec{\#}\right)\right)(r)\right)$. ■

Corollary 1.21.1 is the "getting determinacy" direction of Theorem 1.21 for the case $\beta = 0$. (i) and (ii) are the case of (iii)/(v) and (iv)/(vi) in which $\vec{\#} = \mathcal{S}^{\gamma}_1(\emptyset)$.

Remark 1.22. I wish I could report that the proof of Corollary 1.21.1 is by induction on n. No problem arises with the base step. However, to show Corollary 1.21.1 for $n = m{+}1$, i.e. for $\vec{\#} = \vec{\mathcal{S}}^{\gamma(m+1)}_{k(m+1)} \vec{\mathcal{S}}^{\gamma(m)}_{k(m)} \cdots \vec{\mathcal{S}}^{\gamma(1)}_{k(1)} \vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset) \neq \emptyset$, we use a boldface analogue of the anticipated induction hypothesis, that is, of Corollary 1.21.1(iii) for $\vec{\#} = \vec{\mathcal{S}}^{\gamma(m)}_{k(m)} \vec{\mathcal{S}}^{\gamma(m-1)}_{k(m-1)} \cdots \vec{\mathcal{S}}^{\gamma(1)}_{k(1)} \vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset) \neq \emptyset$. Setting $\vec{\#} = \vec{\mathcal{S}}^{\gamma(m)}_{k(m)} \vec{\mathcal{S}}^{\gamma(m-1)}_{k(m-1)} \cdots \vec{\mathcal{S}}^{\gamma(1)}_{k(1)} \vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset) \neq \emptyset$, such a boldface analogue is that every boldface $(\Gamma_r(\vec{\#}))^{**}$ game with a code (r, T, S^1, S^2) in $L(r)[\vec{\mathcal{S}}^{\gamma(m+1)}_{k(m+1)} \vec{\#}]$ has a w.s. in $L(r)[\vec{\mathcal{S}}^{\gamma(m+1)}_{k(m+1)} \vec{\#}]$.

Corollary 1.21.1 is obtained by showing its boldface analogue (there is more than one acceptable analogue). We show the boldface analogue by first not only fixing r and γ but also fixing the model $M = L(r)[\vec{\mathcal{S}}^{\gamma(\hat{n})}_{k(\hat{n})} \vec{\mathcal{S}}^{\gamma(\hat{n}-1)}_{k(\hat{n}-1)} \cdots \vec{\mathcal{S}}^{\gamma(1)}_{k(1)} \vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset)]$[4] and then showing by induction on n that for $n \leq \hat{n}$, every boldface $\left(\Gamma_r\left(\vec{\mathcal{S}}^{\gamma(n)}_{k(n)} \vec{\mathcal{S}}^{\gamma(n-1}_{k(n-1)} \cdots \vec{\mathcal{S}}^{\gamma(1)}_{k(1)} \vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset)\right)\right)^{**}$ game with code in M has a w.s. in M.

In Section 3, we show Corollary 1.21.1 for $k(n) = 1$, assuming an appropriate boldface analogue of Corollary 1.21.1 for $k(n) > 1$ (see (HYP) of Section 3). We made this analogue slightly weaker than what is suggested in the preceding paragraph.

Remark 1.23. Section 2 is devoted to proving Theorem 1.21 for $k(n) = 1$, assuming the appropriate hypothesis. I wish I could report that the proof is by induction on β. Let $k(n) = 1$. As indicated in Remark 1.22, Section 3 takes care of Corollary 1.21.1, i.e. of Theorem 1.21 for the case $\beta = 0$. We prove Theorem 1.21(iii) for $k(n) = 1$ in Section 2, assuming that Theorem 1.21(iv) holds for β less than the β fixed in (iii) and assuming that Corollary 1.21.1(v) holds with r varying through the reals of the model given in (iii), i.e of the model $L\left(\left(\beta\mathcal{S}_1\left(\vec{\#}\right)\right)(r)\right)\left[\vec{\#}\right]$. (When we invoke Corollary 1.21.1, its real is likely to be different from the fixed r of Theorem 1.21(iii).)

Theorem 1.21(iv) is more difficult. As the reader may have anticipated from Remark 1.22, to show Theorem 1.21(iv) for $k(n) = 1$, we use the following:

- a boldface analogue of Corollary 1.21.1(iii). e.g. setting $M = L\left(\beta\mathcal{S}_1\left(\vec{\mathcal{S}}^{\gamma(n)}_{k(n)} \vec{\#}\right)\right)[\vec{\mathcal{S}}^{\gamma(n)}_{k(n)} \vec{\#}]$ and $\vec{\#} = \vec{\mathcal{S}}^{\gamma(n-1)}_{k(n-1)} \cdots \vec{\mathcal{S}}^{\gamma(1)}_{k(1)} \vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset)$, every boldface $\left(\Gamma_r(\vec{\#})\right)^{**}$ game with code in M has a w.s. in M.
- Theorem 1.21(iii) for β smaller than that fixed, to obtain w.s. for $(\Gamma_r(\vec{\#}), \gamma(n) * \Pi^0_1, < \beta * \Sigma^0_1)^*(r)$ games.
- Corollary 1.21.1 to obtain w.s. for $(\Gamma_r(\vec{\#}), < \gamma * \Pi^0_1)^*_+(x)$ games where x varies through the reals of M. (We may use either part (v) or (vi) of Corollary 1.21.1 since $(\Gamma_r(\vec{\#}), < \gamma(n) * \Pi^0_1)^*_+(x) \subset (\Gamma_r(\vec{\#}), \gamma(n) * \Pi^0_1)^*(x)$.)

§1.3. **Borel Auxiliary Moves**

Martin introduced pairs of Borel Auxiliary Moves in his proof of Borel Determinacy [Ma75].[5] Such pairs

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[4] $\hat{n}$ is some fixed natural number, but we are not performing any induction on $\hat{n}$.

[5] Martin [Ma85] later reorganized his proof, to show that all Borel sets can be unraveled, a condition

are used to transcribe a closed condition of the orginal game into a clopen condition of the auxiliary game, but at the expense of playing higher type objects in the auxiliary game. In [Mo80], Moshovakis presents a proof of Borel determinacy for the finite levels, playing in the auxiliary game slightly different pairs of Borel auxiliary moves (and proving the result in game trees with terminal nodes). In this paper, we use the pairs given in [Mo80].

Such pairs greatly affect the game tree for the auxiliary game: failure of any of the additional requirements corresponding to playing Borel auxiliary moves will be known at some position in the game. This section is my attempt to list such requirements one time instead of repeating the same requirements in full detail in Definitions 2.6, 2.10, and 3.3 the auxiliary games. The inflexibility resulting from this presentation does cause a slight problem as indicated below in Remark 1.27. The reader not familiar with a proof of Borel Determinacy may want to consider this section when encountering Definitions 2.6, 2.10, and 3.3.

A set T of positions of a game G is called an *I-imposed subgame* of G if T is a set of positions of G such that $p \in T$ for any position p of G with even length and which extends some $p' \in T$ of length $\ell h(p) - 1$. Thus, an I-imposed subgame of G is a set of positions which restricts I's moves but does not restrict II's moves.

Suppose for a fixed game G, $\langle Q_i | i \in I \rangle$ is a sequence of I-imposed subgames of G and $\langle p_j | j \in J \rangle$ is a sequence of positions in G. Then $G(\langle Q_i | i \in I \rangle ; \langle p_j | j \in J \rangle)$ is basically the same as the game G, except its game tree consists only of positions "consistent" with each Q_i and p_j. More precisely, both players are required to play so that each position $p = (x(0), x(1), ..., x(n))$ of $G(\langle Q_i | i \in I \rangle ; \langle p_j | j \in J \rangle)$ is in each Q_i and is compatible with each p_j. The player first to fail to meet these requirements loses, and if neither player loses by failing to meet these requirements, then the winning conditions are the same as those for the game G.

$G\left(p_1, p_2, \cdots, p_n\right)$ denotes the game $G(\emptyset ; \langle p_j | 1 \leq j \leq n \rangle)$. We shall sometimes refer to an I-imposed subgame of $G\left(p_1, p_2, \cdots, p_n\right)$ as an I-imposed subgame of G which is consistent with $p_1, p_2, \cdots, p_n$ (even though such a subgame may not be a I-imposed subgame of G).

As indicated above, the auxiliary games which we encounter in Sections 2 and 3 include (pairs of) Borel auxiliary moves (and designated moves $x\left(i\right)$ for $i \in \omega$ which are to be thought of as corresponding to the moves of original game). The Borel auxiliary moves are used to specify a particular subgame of the original game for the designated play x to proceed along. For our purposes here, the $x\left(i\right)$ will be integer moves.

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which implies the determinacy of the given set. It is well-known that unraveling $\mathbf{\Pi^1_1}$ sets is much stronger than the determinacy of $\mathbf{\Pi^1_1}$ games. An important open problem is to prove that all $\mathbf{\Pi^1_1}$ sets can be unraveled from some large cardinal or determinacy hypotheses.

Definition 1.24. Borel auxiliary moves. Let G be a game with integer moves $x(i)$ for $i \in \omega$. Let G^* be a game with: designated integer moves $x(i)$ for $i < \omega$, a pair of auxiliary moves T and $\langle \hat{t}, t \rangle$, and possibly some other moves. Assume that in both G and G^*:

- $x(i)$ precedes the play of $x(j)$ if $i < j$.
- I plays $x(i)$ iff i is even.

Then T and $\langle \hat{t}, t \rangle$ are *Borel auxiliary moves* of G^* with respect to G if:

(1) Player I plays T and the next move of II is $\langle \hat{t}, t \rangle$.

(2) If $\langle x(i) \,|\, i < k \rangle$ and $\langle Q_i; \langle \hat{q}_i, q_i \rangle \,|\, i < n \rangle$ respectively are the sequences of integer moves $x(i)$ and Borel auxiliary moves which precede the play of T in G^*, then T is an I-imposed subgame of the game $G(\langle Q_i | \hat{q}_i = 1 \rangle ; \langle q_i | \hat{q}_i = 0 \rangle , \bar{x}(k))$.

(3) $\langle \hat{t}, t \rangle \in \{ \langle 1, - \rangle , \langle 0, q \rangle \,|\, q \in T$ and q has odd length $\}$.

(4) If II plays $\hat{t} = 0$, then both players may only play integer moves $x(i-1)$ so that $\bar{x}(i)$ and t are compatible. If II plays $\hat{t} = 1$, then both players are required to play $x(i-1)$ so that $\bar{x}(i) \in T$–of course this is a requirement for player I since T is I-imposed. $\triangle$

Definition 1.25. Let G and G^* be as in Definition 1.24. Suppose T and $\langle \hat{t}, t \rangle$ are a pair of Borel auxiliary moves of an auxiliary game G^* with respect to G. We say that $R \subseteq \omega$ and G *determines* the Borel auxiliary moves T and $\langle \hat{t}, t \rangle$ of G^* if the following hold in addition to (1) through (4) of Definition 1.24:

(5) If $\hat{t} = 0$, then $R(t)$.

(6) If the integer moves $x(0), x(1), x(2), \cdots, x(i-1)$ precede the play of move T and $R(\bar{x}(i))$, then II must play so that $\hat{t} = 0$.

(7) If Borel auxiliary moves Q and $\langle \hat{q}, q \rangle = \langle 0, \bar{x}(2i-1) \rangle$ precede the play of the move T and $\exists j < 2i R(\bar{x}(j))$, then II must play so that $\hat{t} = 0$.

(8) If II plays $\hat{t} = 1$, then II loses the game G^* once an integer move $x(i-1)$ is played such that $\neg R(\bar{x}(i))$. $\triangle$

We shall say (in the above case) that the Borel auxiliary moves T and $\langle \hat{t}, t \rangle$ are determined by R when G and G^* are clear from the context. R is typically related to the payoff set of the original game G in such a way that we can transcribe the winning conditions for G into simpler winning conditions for the auxiliary game G^* (but at the expense of introducing higher type objects). Since failure of any of the conditions listed in Definitions 1.24 and 1.25 will occur at some finite position in the auxiliary game G^*, T and $\langle \hat{t}, t \rangle$ being a pair of Borel auxiliary moves determined by $R \subseteq \omega$ implicitly defines the game tree T^* along which G^* is played. For legal plays of G^* (in the situation described in Definition 1.25),

$$\hat{t} = 0 \text{ iff } \exists i \, R(\bar{x}(i)),$$

and so the collection of plays of G^* for which $\exists i\, R(\bar{x}(i))$ is a clopen subset of the game tree T^*. In the auxiliary games we consider, we restrict player I to play the Borel auxiliary move T from a certain model, often so that we have indiscernibles for $L\,[T]$ (from the hypotheses under consideration).

Given an integer game A and a pair of Borel auxiliary moves T and $\langle \hat{t}, t\rangle$ determined by $R \subseteq \omega$, we next define $A(R;T;\langle \hat{t}, t\rangle)$ as the game:

- to be played on the game tree implicitly defined by Definitions 1.24 and 1.25
- and having the same payoff as the game A for legal plays of $A(R;T;\langle \hat{t}, t\rangle)$, i.e., if y is a legal play of the game $A(R;T;\langle \hat{t}, t\rangle)$ and x is the corresponding play of integer moves, then
$$y \in A(R;T;\langle \hat{t}, t\rangle) \Leftrightarrow x \in A.$$

Definition 1.26. Let $R \subseteq \omega$. Let T and $\left\langle \hat{t}, t\right\rangle$ be a pair of Borel auxiliary moves determined by R. Let A be a set of reals.

1. $A\,(R;T)$. If $R\,(\bar{x}\,(i))$ or $\bar{x}\,(i) \notin T$ for some least i, then: $x \in A\,(R;T) \Leftrightarrow \bar{x}\,(i) \in T$. If $\forall i\,(\neg R\,(\bar{x}\,(i))$ and $\bar{x}\,(i) \in T)$, then: $x \in A\,(R;T) \Leftrightarrow x \in A$.
2. $A\,(t)$. If $\bar{x}\,(i)$ is not compatible with t for some least i, then: $x \in A\,(t) \Leftrightarrow i$ is even.[6] If x extends t, then: $x \in A\,(t) \Leftrightarrow x \in A$.
3. $A\left(R;T;\left\langle \hat{t}, t\right\rangle\right)$. Let $A\,(R;T;\langle 1, -\rangle) =_{df} A\,(R;T)$. If $t \in T$ and $R\,(t)$, then $A\,(R;T;\langle 0, t\rangle) = A\,(t)$. (For completeness, if $t \notin T$ or $\neg R\,(t)$, then set $A\,(R;T;\langle 0, t\rangle = {}^{\omega}\omega$; this last case does not occur by Definition 1.24 of a pair of Borel auxiliary moves.) $\triangle$

Remark 1.27. Suppose $R \subseteq \omega$ determines a pair T and $\langle \hat{t}, t\rangle$ of Borel auxiliary moves of G^* with respect to an integer game A. The setting for such a pair presumes that integer moves $x(i)$ are played in G^* for each $i \in \omega$. We shall consider auxiliary games for the game A in which such T and $\langle \hat{t}, t\rangle$ are played, but following the play of $\langle \hat{t}, t\rangle$, instead of playing the remaining integer moves, player I is required to have a w.s. for what amounts to the remainder of the game A along the game tree implicitly given by the Borel auxiliary moves. For instance, in the auxiliary game $G^{\gamma\beta}$ of Definition 2.6 for a particular integer game E, following the play of T and $\langle \hat{t}, t\rangle$, no integer moves need be played and player I must have a w.s. for the game $E(R;T;\langle \hat{t}, t\rangle)$–note that conditions (4) and (8) of Definition 1.25 are transcribed into the Definition 1.26 of $E(R;T;\langle \hat{t}, t\rangle)$.

DO WE NEED DEFINITIONS 0.18 PAGE 29 OF 4ba2.tex or THE DISCUSSION PRECEDING IT?

§1.4 **Ordinal Auxiliary Moves.**

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[6] A play x is a loss for I in the game $A\,(t)$ if I plays a move $x\,(2j)$ which is first to witness x being incompatible with t.

We shall need ordinal auxiliary moves in the auxiliary games $G_{+}^{\gamma\beta}$ and G_{+}^{γ}, respectively presented in Definitions 2.10 and 3.3. The reader may wish to consider this section when he/she encounters these games. $G_{+}^{\gamma\beta}$ and G_{+}^{γ} are the auxiliary games corresponding to the $(\cdots)_{+}^{*}$ games of this paper. The auxiliary games presented in this paper for the $(\cdots)^{*}$ games do not have such moves; however, the use of such moves is hidden by our induction hypotheses, and the auxiliary games used in our proofs of determinacy for both types of classes in [Du5] (of some of the same results) include such moves.

In $G_{+}^{\gamma\beta}$ and G_{+}^{γ}, we have the usual set-up involving ordinal auxiliary moves. In this set-up, designated integer moves $x(i)$ and ordinal auxiliary moves ξ_i are played so that:

i.) If $i<j$, then $x(i)$ is played before $x(j)$ and ξ_i is played before ξ_j.

ii.) If $x(i)$ or ξ_i is played by I [respectively II], then i is even [resp. odd]. We shall prefer to think of ξ_i as some particular ξ_j^{α} $(j<\omega, \alpha<\omega\cdot m$ for some fixed $m<\omega)$ by setting $\xi_j^{\alpha}=\xi_{\pi_m(\alpha,j)}$ for an appropriate π_m.

For each $m<\omega$, select a recursive bijection $\pi_m\colon\ \omega\cdot m\times\omega\longrightarrow\omega$, which satisfies the conditions below in parentheses, and then the ξ_j^{α} satisfy the following:

- ξ_j^{α} is played before ξ_{j+1}^{α} $\Big(\pi_m(\alpha,j)<\pi_m(\alpha,j+1)\Big)$.
- $\xi_0^{\omega\cdot n+k}$ is played before $\xi_0^{\omega\cdot n+k+1}$ $\Big(\pi_m(\omega\cdot n+k,0)<\pi_m(\omega\cdot n+k+1,0)\Big)$.
- ξ_j^{α} is played by $I\Leftrightarrow\alpha$ is even $\Big(\pi_m(\alpha,j)$ is even $\Leftrightarrow\alpha$ is even$\Big)$.

A real $x\in{}^{\omega}\omega$ is in a fixed $\Pi_1^1(z)$ set A_α iff a certain ordering $<_x^{\alpha}$ is a wellordering. For the real x of integer moves ($x(i)=i^{\text{th}}$ integer move), ξ_i^{α} and $<_x^{\alpha}$ will be used to verify, when true, that $x\in A_\alpha$. Lemma 1.28 provides such orderings.

Lemma 1.28. (Kleene [Kl55]) Let $\beta<\omega_1^{CK}(z)$ and let $\langle A_\alpha|\alpha<\beta\rangle$ witness $A\in\beta\text{–}\Pi_1^1(z)$ so that there exists a recursive-in-z wellordering of a subset E of ω with order type β such that $\big\{(k,x)\in E\times({}^{\omega}\omega)\,|x\in A_{|k|}\big\}\in\Pi_1^1(z)$. Then there exists a function F recursive in z and with domain $E\times\{\bar{x}(i)\,|x\in{}^{\omega}\omega$ and $i\in\omega\}$ such that:

(1) $<_{\bar{x}(i)}^{|n|}=_{df}F(n,\bar{x}(i))$ is a linear ordering of $0,1,2,\cdots,i$ with largest element 0.

(2) $<_{\bar{x}(i)}^{|n|}$ is a subordering of $<_{\bar{x}(j)}^{|n|}$ if $i\le j$.

(3) $x\in A_{|n|}$ iff $<_x^{|n|}=_{df}\bigcup\limits_{i\in\omega}<_{\bar{x}(i)}^{|n|}$ is a wellordering. ■

If the map $j\mapsto\xi_j^{\alpha}$ from ω under the ordering $<_x^{\alpha}$ and into the ordinals is order-preserving, then $<_x^{\alpha}$ is a wellordering and, by Lemma 1.28(iii), $x\in A_\alpha$. Using ordinal auxiliary moves to construct such order preserving maps, Martin proved in the early 1970's the following:

Theorem 0.8 of [Du90]. (Martin's Theorem) [Ma∞] $\text{Det}\big(<\omega^2-\Pi_1^1(z)\big)$ follows from the existence of indiscernibles for $L(z)$. ■

We give the proof of Martin's Theorem for the case $z = \emptyset$ in detail in Theorem 0.8 of [Du90], and of course Martin's upcoming book [Martin∞] on determinacy presents this result.

Let $\langle A_\alpha | \alpha < \omega \cdot m \rangle$ witness that a fixed set A is $\omega \cdot m - \Pi^1_1(z)$ (for some $m < \omega$). In the proof of Theorem 0.8, an open auxiliary game G', corresponding to $G(A)$, is defined. Corresponding to each $\Pi^1_1(z)$ set A_α ($\alpha < \omega \cdot m$), a sequence ξ^α_j ($j \in \omega$) of ordinal auxiliary moves is played in G'.

Using indiscernibles for $L(z)$, a w.s. for $G(A)$ is constructed by integrating a w.s. for G' with respect to the ordinal auxiliary moves ξ^α_j. Using the existence of indiscernibles for the appropriate model, we shall similarly integrate, with respect to ordinal auxiliary moves, winning strategies for the auxiliary games in Sections 2 and 3. To carry out such integrations, the ξ^α_j must be appropriately ordered and certain limitations are placed on the size of ξ^α_j in terms of α.

Definition 1.29. Let $\left(x(0), x(1), x(2), \cdots, x\left(\hat{i}-1\right)\right)$ and $\left(\xi_0, \xi_1, \xi_2, \cdots, \xi_{\hat{j}-1}\right)$ be sequences, respectively, of integers and of ordinals. Let $\langle A_\alpha | \alpha < \omega \cdot m \rangle$ witness that some set is $\omega \cdot m - \Pi^1_1(z)$. Let F and $<^\alpha_x$ be as in Lemma 1.28 for the case $\beta = \omega \cdot m$, so that:

$$x \in A_\alpha \text{ iff } <^\alpha_x \text{ is a wellordering.}$$

Recall π_m defined in the second paragraph of this section §1.4. Whenever $\pi_\alpha(j) < \hat{j}$, let $\xi^\alpha_j = \xi_{\pi_\alpha(j)}$; in this case, $\left(\xi_0, \xi_1, \xi_2, \cdots, \xi_{\hat{j}-1}\right)$ includes $\left(\xi^\alpha_0, \xi^\alpha_1, \xi^\alpha_2, \cdots, \xi^\alpha_j\right) = \left(\xi_{\pi_\alpha(0)}, \xi_{\pi_\alpha(1)}, \xi_{\pi_\alpha(2)}, \cdots, \xi_{\pi_\alpha(j)}\right)$ since π_α is an increasing function.

We say that $\left(\xi_0, \xi_1, \xi_2, \cdots, \xi_{\hat{j}-1}\right)$ is *properly ordered* with respect to $\left(x(0), x(1), x(2), \cdots, x\left(\hat{i}-1\right)\right)$ (or just $\bar{x}\left(\hat{i}\right)$) and $\langle A_\alpha | \alpha < \omega \cdot m \rangle$ using $\langle \kappa_{n+1} | n < \omega \rangle$ iff whenever $\pi_\alpha(j) < \hat{j}$:

(i) $\xi^\alpha_j \in \kappa_{n+1}$ if $\alpha = \omega \cdot n + k$ for some $k < \omega$ and $n < m$.

(ii) The map from the field of $<^\alpha_{(\bar{x}(\hat{i}))}$ into the ordinals defined by $k \hookrightarrow \xi^\alpha_k$ is order-preserving.

In this case, if p is a position of a game G whose designated integer moves $x(i)$ and ordinal auxiliary moves ξ_j are exactly $x(0), x(1), x(2), \cdots, x\left(\hat{i}-1\right)$ and $\xi_0, \xi_1, \xi_2, \cdots, \xi_{\hat{j}-1}$, then we shall say that position p is *properly ordered* with respect to the ξ_j's, the $x(i)$'s, and $\langle A_\alpha | \alpha < \omega \cdot m \rangle$ in G using $\langle \kappa_{n+1} | n < m \rangle$. Typically, G, $\langle A_\alpha | \alpha < \omega \cdot m \rangle$, and the designated integer moves $x(i)$ are clear from the context and so we suppress these. For a fixed game G, we shall say the ξ_j's are *properly ordered* using $\langle \kappa_{n+1} | n < m \rangle$ if every position of G is properly ordered with respect to the ξ_j's using $\langle \kappa_{n+1} | n < m \rangle$. $\triangle$

In this paper, our auxiliary games include at most one sequence of ordinal auxiliary moves and so it suffices to take $\langle \kappa_{n+1} | n < m \rangle$ to be $\langle \omega_{n+1} | n < m \rangle$. Our auxiliary games having at most one such sequences is made possible by our proving the results of this paper inductively. When proving our results directly from the corresponding open auxiliary games as in [Du5], we consider several such sequences of ordinal auxiliary moves and the situation appears slightly different.