**Section 2(continued).**

Next we wish to show that:

(xiv) If $(\beta+1)\#_{\gamma+1}(r)$ exists, then $E_+$ has a w.s. $s_+ \in M^\#$.

We would prefer to do this by asking the reader to make slight changes to the proof of (i), i.e. by carrying out the following procedure:

(xv) Define an auxiliary game $\hat{G}_+^{\gamma,\beta}$ analogous to $G^{\gamma,\beta}$.

(xvi) Prove that $M^\# = L((\beta+1)\#_{\gamma+1}(r))$ has a w.s. $\hat{S}^+_{(T;\langle\hat{t},t\rangle)}$ for $E_+\left(R;T;\langle\hat{t},t\rangle\right)$ for appropriate $T$ and $\langle\hat{t},t\rangle$.

(xvii) Prove the analogues of Lemmas 2.8 and 2.9, which I shall respectively call Lemmas $\hat{1}$ and $\hat{2}$.

This procedure essentially works for showing Lemma $\hat{1}$: Suppose $\hat{G}_+^{\gamma,\beta}$ has the same moves as $G^{\gamma,\beta}$ but $E_+(R;T;\langle\hat{t},t\rangle)$ replaces $E(R;T;\langle\hat{t},t\rangle)$ in (xii). By Proposition 2.5, $S^+_{(T;\langle 0,t\rangle)} \in M$ and $S^+_{(T;\langle 1,-\rangle)} \in L(\#_{\gamma+1}(r,T)) \subseteq M^\#$ so that:

$$\text{player I has a w.s. for } \hat{G}_+^{\gamma,\beta} \text{ iff he has one in } M^\#.$$

If player I has a w.s. for $\hat{G}_+^{\gamma,\beta}$, then a w.s. $s_+ \in M^\#$ of player I for the game $E_+$ is constructed analogous to $s$ of Lemma 2.8, substituting $S^{\gamma,\beta}$ with a w.s. $\hat{S}_+^{\gamma,\beta} \in M^\#$ for $\hat{G}_+^{\gamma,\beta}$ and each $S_{(T;\langle\hat{t},t\rangle)}$ with $S^+_{(T;\langle\hat{t},t\rangle)}$. If $x$ is a play consistent with $s_+$, show $x \in E_+$ by the argument analogous to that of the Claim in Lemma 2.8, substituting $E$ with $E_+$, $E(R;T;\langle\hat{t},t\rangle)$ with $E_+(R;T;\langle\hat{t},t\rangle)$, and $S_{(T;\langle\hat{t},t\rangle)}$ with $S^+_{(T;\langle\hat{t},t\rangle)}$.

However, the procedure (outlined in (xiv) through (xvii)) fails for showing Lemma $\hat{2}$, i.e. fails to provide II with a w.s. $s_+ \in M^\#$ for the game $E_+$ (under the appropriate hypothesis). Its failure though helps to reveal that the complexity of $T$ in Lemmas 2.9 and 2.14 is the key to the proofs of (i) and (ii).

The question of whether we should require $T \in M$ or $T \in M^\#$ becomes the primary problem when $\hat{t} = 1$.[13] Since $E_+(R;T) \in \left(\Gamma, \gamma * \Pi^0_1\right)^*_+(r,R,T)$ and $R \in L(r)$, $\hat{S}^+_{(T;\langle 1,-\rangle)} \in L(\#_{\gamma+1}(r,T))$. For (xvi) to hold, we need $T \in M$; for then

(xviii) $\#_{\gamma+1}(r,T) \in M^\# = L((\beta+1)\#_{\gamma+1}(r))$ so that $\hat{S}^+_{(T;\langle 1,-\rangle)} \in L(\#_{\gamma+1}(r,T)) \subseteq L((\beta+1)\#_{\gamma+1}(r)) = M^\#$.

So let us require that $T \in M$ but replace (xii) with:

$$\text{I wins } \hat{G}_+^{\gamma,\beta} \Leftrightarrow \text{I has a w.s. for } E_+\left(R;T;\langle\hat{t},t\rangle\right).$$

Since $\hat{S}^+_{(T;\langle\hat{t},t\rangle)} \in M^\#$, $\hat{G}_+^{\gamma,\beta}$ has a w.s. $\hat{S}_+^{\gamma,\beta} \in M^\#$. But then $T$, as defined in (xiii) with $\hat{S}_+^{\gamma,\beta}$ replacing $S^{\gamma,\beta}$, is in $M^\#$ and not necessarily in $M$ (as required).

$T$ relies on $\hat{S}_+^{\gamma,\beta}$ which in turn relies on $\hat{S}^+_{\langle 1,-\rangle}$. But to get $\hat{S}^+_{\langle 1,-\rangle}$ we need indiscernibles for an appropriate

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[13] $s^+_{(T;\langle 0,t\rangle)} \in M$ holding follows by induction since $E_+(t) \in \left(\Gamma, \gamma * \Pi^0_1, < \beta * \Sigma^0_1\right)^*_+(r)$.

model which contains $T$. Consequently, to show (xiv) for the case $S_+$ is II's w.s., we will not use a w.s. $\hat{S}^+_{\langle 1,-\rangle}$ for some $E_+(R;T;\langle 1,-\rangle)$–see Lemma 2.14 below.

**Definition 2.10.** $G^{\gamma,\beta}_+$. As with $G^{\gamma,\beta}$, the first two moves of $G^{\gamma,\beta}_+$ are Borel auxiliary moves $T$ and $\langle \hat{t}, t\rangle$ such that (xi) of Definition 2.6 holds (hence $T \in M$). If $\hat{t} = 0$, there is no need to play any more moves. If $\hat{t} = 1$, then the moves of players I and II during the $(2n+1)^{st}$ inning are respectively $x(2n)$, $\lambda_n$ and $x(2n+1)$, $\lambda_{2n+1}$, where the $x(i)$'s are integer moves, the $\lambda_i$'s are ordinal auxiliary moves, and

(xix) the $\lambda_i$'s must be properly ordered with respect to $\vec{D}$ using $\langle \omega_{i+1} | i < m\rangle$.

Typical plays of $G^{\gamma,\beta}_+$ are of the following form:

$$G^{\gamma,\beta}_+ \quad \begin{array}{llllllll} \text{I} & T & x(0),\lambda_0 & & x(2),\lambda_2 & \cdots & x(2i),\lambda_{2i} & \\ \text{II} & \langle 1,-\rangle & & x(1),\lambda_1 & x(3),\lambda_3 & & & x(2i+1),\lambda_{2i+1} \cdots \end{array}$$

$$G^{\gamma,\beta}_+ \quad \begin{array}{ll} \text{I} & T \\ \text{II} & \langle 0,t\rangle \end{array}.$$

The first player to violate (xi) or (xix) loses. If (xi) and (xix) have not been violated, then:

(xx) I wins $G^{\gamma,\beta}_+ \Leftrightarrow$ either $\left(\hat{t} = 1 \text{ and } x \in (B)^*\left(\vec{A}\right)\right)$

or ($\hat{t} = 0$ and I has a w.s. for the game $E_+(R;T;\langle 0,t\rangle) = E(R;T;\langle 0,t\rangle)$).[14] △

**Remark 2.11.** Suppose $M \models$ "$\forall\hat{\gamma} < \gamma\ \#_{\hat{\gamma}+1}$ is total" and $(T;\langle 0,t\rangle)$ is a legal position of the game $G^{\gamma,\beta}_+$. Then $T \in M$ and by Proposition 2.5, the game $E_+(R;T;\langle 0,t\rangle) = E(R;T;\langle 0,t\rangle)$ has a w.s. $S_{(T;\langle 0,t\rangle)} \in M$. Therefore, in this case, for $\lambda = I, II$:

$$\text{player } \lambda \text{ wins } G^{\gamma,\beta}_+ \text{ iff player } \lambda \text{ has a w.s. } S_{(T,\langle 0,t\rangle)} \in M.$$

In Section 3, we show

**Theorem 3.1(i).** If $z$ is a real and $L(z)\left[\vec{\#}_\gamma\right] \models \forall\hat{\gamma} < \gamma\ \#_{\hat{\gamma}+1}$ is total", then every $\left(\Gamma, \gamma * \Pi^0_1\right)^*(z)$ game has a w.s. $S_\gamma \in L(z)\left[\vec{\#}_\gamma\right]$.

By essentially the same proof, we have:

**Corollary 3.1.4.** If $M \models$ "$\forall\hat{\gamma} < \gamma\ \#_{\hat{\gamma}+1}$ is total", then $G^{\gamma,\beta}_+$ has a w.s. $S^{\gamma,\beta}_+ \in M$.

The reader willing to accept Corollary 3.1.4 can proceed to read the rest of this section. Below we outline the proof to Corollary 3.1.4. This outline is based on adjustments to the proof of Theorem 3.1(i); in particular, it is based on Definitions 3.2 and 3.3, and Lemmas 3.8 and 3.9.

**Outline of Proof:** Recall that $B$ is a fixed $(\Gamma, \gamma * \Pi^0_1)(r)$ set throughout this section. Let $C \in \Gamma(r)$ and $\vec{B} = \langle B_\alpha | \alpha < \gamma\rangle$ witness that $B \in (\Gamma, \gamma * \Pi^0_1)(r)$. (This notation is consistent with that used in Section 3 but be careful not to confuse/relate $C$ with any of the components of $\vec{C}$, which was fixed at the start of this section.)

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[14] Of course, player I may win $G^{0,\beta}_+$ for a play in which $\hat{t} = 1$ is played, if for instance II cannot play some ordinal auxiliary move $\lambda_{2i}$.

Assume $M \models \forall \hat{\gamma} < \gamma \ \#_{\hat{\gamma}+1}$ is total. For legal plays (e.g. no $\lambda_i$ is badly played) of $G_+^{\gamma,\beta}$ in which $\hat{t} = 1$:

$$\text{player I wins } G_+^{\gamma,\beta} \Leftrightarrow x \in \left(B^*\left(\vec{A}\right)\right)(R;T;\langle 1,-\rangle).$$

$B^*\left(\vec{A}\right) \in \left(\Gamma, \gamma * \Pi_1^0\right)^*(r)$ and $\left(B^*\left(\vec{A}\right)\right)(R;T;\langle 1,-\rangle) \in \left(\Gamma, \gamma * \Pi_1^0\right)^*(r,R,T)$. By Theorem 3.1,

(xxi) the game $\left(B^*\left(\vec{A}\right)\right)(R;T;\langle 1,-\rangle)$ has a w.s. $S_\gamma \in L(r,T)\left[\vec{\#}_\gamma\right]$, assuming $L(r,T)\left[\vec{\#}_\gamma\right] \models$ "$\forall \hat{\gamma} < \gamma \ \#_{\hat{\gamma}+1}$ is total".

In the proof of (xxi), we:

(xxii) Define an auxiliary game $G^\gamma$ (see Definition 3.3) which includes a sequence $\left\langle T_n; \left\langle \hat{t}_n, t_n\right\rangle | \forall i < n \ \hat{t}_i = 0\right\rangle$ of Borel auxiliary moves.

(xxiii) Note that $G^\gamma$ is boldface $(\mathbf{\Gamma})^{**}$ (open in the case $C = \emptyset$) with code in $L(r,T)[\vec{\#}_\gamma]$ and that $G^\gamma$ has w.s. $S_\gamma \in L(r,T)\left[\vec{\#}_\gamma\right]$.

(xxiv) Simulate the Borel auxiliary moves $T_i$ and $\left\langle \hat{t}_i, t_i\right\rangle$ in the usual way to define from $S_\gamma$, a w.s. $\sigma \in L(r,T)\left[\vec{\#}_\gamma\right]$ for $\left(B^*\left(\vec{A}\right)\right)(R;T;\langle 1,-\rangle)$.

Analogous to $G^\gamma$, define an auxiliary game $\left(G_+^{\gamma,\beta}\right)^\gamma$ for $G_+^{\gamma,\beta}$, where $\left(G_+^{\gamma,\beta}\right)^\gamma$ contains a sequence $\left\langle T_n; \left\langle \hat{t}_n, t_n\right\rangle | \forall i < n \ \hat{t}_i = 0\right\rangle$ of Borel auxiliary moves when $\hat{t} = 1$. $\left(G_+^{\gamma,\beta}\right)^\gamma$ and $G_+^{\gamma,\beta}$ are identical for plays in which $\hat{t} = 0$. Typical plays of $(G^{\gamma,\beta})^\gamma$ have the following form:

(1) $(G_+^{\gamma,\beta})^\gamma \quad \begin{matrix} \text{I} \\ \text{II} \end{matrix} \quad \begin{matrix} T & \\ & \langle 0,t\rangle \end{matrix}.$

(2) $(G_+^{\gamma,\beta})^\gamma \quad \begin{matrix} \text{I} \\ \text{II} \end{matrix} \quad \begin{matrix} T & & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & & \cdots & T_{n-1} & & x(2n-2),\lambda_{2n-2} & & T_n & \\ & \langle 1,-\rangle & & \langle 0,t_0\rangle & & x(1),\lambda_1 & & \langle 0,t_1\rangle & & x(3),\lambda_3 & & & \langle 0,t_{n-1}\rangle & & x(2n-1),\lambda_{2n-1} & & \langle 1,-\rangle \end{matrix}.$

(3) $(G_+^{\gamma,\beta})^\gamma \quad \begin{matrix} \text{I} \\ \text{II} \end{matrix} \quad \begin{matrix} T & & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & & \cdots & T_n & & x(2n),\lambda_{2n} & & \cdots \\ & \langle 1,-\rangle & & \langle 0,t_0\rangle & & x(1),\lambda_1 & & \langle 0,t_1\rangle & & x(3),\lambda_3 & & & \langle 0,t_n\rangle & & x(2n+1),\lambda_{2n+1} & \end{matrix}.$

We require $T_i \in L(r,T)[\vec{\#}_\gamma]$ and $T_i$ to be a I-imposed subgame of $T$. Analogous to the winning conditions for $G^\gamma$ (see Definition 3.3), we have the following winning conditions for legal plays of $(G_+^{\gamma,\beta})^\gamma$:

I wins $(G_+^{\gamma,\beta})^\gamma$ iff one of the following hold:

(1) $\hat{t} = 0$ and player I has a w.s. for the game $E_+(R;T;\langle 0,t\rangle) = E(R;T;\langle 0,t\rangle)$.

(2) $\hat{t} = 1$, $\exists n \hat{t}_n = 1$, and player I has a w.s. for the game $\left[\left(C, (\vec{B})_{(n)_0}\right)^*\left(\vec{A}, dk(\vec{A}_{\omega\cdot(n)_1})\right)\right]\left(R_n; T_n; \bar{x}(i_{max}(2n))\right)$– the notation for this game is given in Definition 3.2.

(3) $\hat{t} = 1$, $\forall n \, \hat{t}_n = 0$, and $x \in (C)^*(\vec{A})$.

Note that (1) gives the same winning conditions as for $G_+^{\gamma,\beta}$ when $\hat{t} = 0$, whereas when $\hat{t} = 1$, (2) and (3) give the same winning conditions as for $G_+^\gamma$ of Definition 3.3.

$(G_+^{\gamma,\beta})^\gamma$ is a boldface $(\mathbf{\Gamma})^{**}$ game with code in $M$ and has a w.s. $\sigma \in M$ (see Remark 3.4). Simulating the Borel auxiliary moves $T_i, \left\langle \hat{t}_i, t_i\right\rangle$ in the usual way (see Lemmas 3.8 and 3.9), we define from $\sigma$, a w.s. $S_+^{\gamma,\beta} \in M$. ∎

We shall integrate $S_+^{\gamma,\beta}$ with respect to the $\lambda_i$'s when proving Lemma 2.12, the analogue of Lemma 2.9.

**Lemma 2.12.** If $(\beta+1)\,\#_{\gamma+1}(r)$ exists and $S_+^{\gamma,\beta}$ is a w.s. for I, then player I has a w.s. $s_+ \in L\left((\beta+1)\,\#_{\gamma+1}(r)\right)$ for the game $\left(B,\vec{C}\right)^*\left(\vec{A},D\right)$.

**Proof:** Assume $S_+^{\gamma,\beta} \in M$ is a w.s. for I and there exists a class $C$ of indiscernibles for $M$, closed and unbounded beneath every uncountable cardinal.

**Construction of $s_+$:** Except for having to simulate playing the ordinal auxiliary moves $\lambda_{2i}$'s, we construct $s_+$ analogous to $s$ of Lemma 2.8, substituting $S^{\gamma,\beta}$ with $S_+^{\gamma,\beta}$ and (each) $S_{(T;\langle\hat{t},t\rangle)}$ with $S_{(T;\langle\hat{t},t\rangle)}$.

Suppose $p = \Big(x(0);x(1);x(2);...;x(2n{-}1)\Big)$ is a position consistent with $s_+$ such that $\forall j \le 2n\,\neg R(\bar{x}(j))$. Pick

$$\lambda_1(n), \lambda_3(n), \lambda_5(n), ..., \lambda_{2n-1}(n) \in C$$

so that for each even $\alpha$, the (partial) map from the field on $<^{\alpha}_{\bar{x}(2n)}$ into the ordinals is order preserving. Then there exist $\lambda_0(n), \lambda_2(n), \lambda_4(n), ..., \lambda_{2n}(n)$, $\tilde{x}(0), \tilde{x}(2), \tilde{x}(4), ..., \tilde{x}(2n-2)$, and $x(2n)$ such that

$$q_n = \Big(T;\langle 1,-\rangle;\tilde{x}(0),\lambda_0(n);\tilde{x}(1),\lambda_1(n);\tilde{x}(2),\lambda_2(n);...;\tilde{x}(2n),\lambda_{2n}(n)\Big)$$

is consistent with $S_+^{\gamma,\beta}$. Let $s_+(p) =_{df} x(2n)$. Note that each $\tilde{x}(2i) = x(2i)$ since $S^+_{(T;\langle 1,-\rangle)} \in M$ by Corollary 3.1.4 and since the $\lambda_{2j-1}(i)$ $(i \le n,\ j \le i)$ are selected from indiscernibles for $M$.

For the case in which we reach a position $\bar{x}(i)$ such that $R(\bar{x}(i))$ and $\forall j < i \neg R(\bar{x}(j))$, the construction of $s_+$ is the same as that of $s$ in Lemma 2.8: By (v), $i$ is odd. Since $S_+^{\gamma,\beta}$ is a w.s. for player I and by Proposition 2.5, $S^+_{(T;\langle 0,\bar{x}(i)\rangle)} \in M$ is a w.s. for player I. Let $s_+(p) =_{df} S^+_{(T;\langle 0,\bar{x}(i)\rangle)}(p)$ for any position $p$ which is compatible with $\bar{x}(i)$.

Note that $s_+ \in M^{\#}$.

**Claim.** $s_+$ is a w.s. for I in the game $\left(B,\vec{C}\right)^*\left(\vec{A},D\right)$.

Let $x$ be a play of the game $E_+$ which is consistent with $s_+$. We show $x \in E_+$. If $\exists i R(\bar{x}(i))$, then $x \in E \subseteq E_+$ by the argument given in the Claim of Lemma 2.8. So assume $\forall i \neg R(\bar{x}(i))$. We consider the two subcases depending on whether $\exists n B(x,n)$.

Suppose $\exists n B(x,n)$. By (iii), $x \in \bigcap_{\alpha<\omega\cdot m} D_\alpha$ so that there exists a sequence $\langle\lambda_{2i}|i<\omega\rangle$ properly ordered with respect to $x$ and $\langle D_\alpha|\alpha \text{ even}, \alpha<\omega\cdot m\rangle$, and such that each $\lambda_{2i} \in C$. Then $\langle\tilde{x}(2i+1),\lambda_{2i+1}|i<\omega\rangle$ exists such that the play

$$y =_{df} \Big(T;\langle 1,-\rangle;x(0),\lambda_0;\tilde{x}(1),\lambda_1;x(2),\lambda_2;\tilde{x}(3),\lambda_3;...;x(2i),\lambda_{2i};\tilde{x}(2i+1),\lambda_{2i+1};...\Big)$$

is consistent with player I's w.s. $S_+^{\gamma,\beta}$. $\tilde{x}(2i+1) = x(2i+1)$ for $i \in \omega$ since each $\lambda_{2i} \in C$ and $S^+_{(T;\langle 1,-\rangle)} \in M$. Since $y$ is consistent with $S_+^{\gamma,\beta}$, $x \in B^*(\vec{A}) \subseteq E_+$.

Now suppose $\forall n \neg B(x,n)$. Then, since we are also in the case $\forall i \neg R(\bar{x}(i))$,

$$x \in E_+ \Leftrightarrow x \in D.$$

Using indiscernibles for $M$, integrate player I's w.s. $S_+^{\gamma,\beta}$ with respect to the $\lambda_i$'s to get $x \in D$ (see Remark

2.13 for more details). Thus, $x \in E_+$. ∎

**Remark 2.13.** At the end of the proof above (to the Claim of Lemma 2.12), we ask the reader to integrate $S_+^{\gamma,\beta}$ with respect to the $\lambda_i$'s to get $x \in D$. The reader not familiar with such integration of strategies should review the proof of Martin's Theorem, stated as Theorem 0.8 of [Du90] in §1.4 of this paper. In this remark, we try to provide an outline of the integration.

For even $\beta < \omega \cdot m$, one shows that

(xxv) if $x \in \bigcap_{\alpha<\beta} D_\alpha$, then $x \in D_\beta$.

For odd $\alpha < \beta$, since $x \in D_\alpha$, there exist $\hat{\lambda}_0^\alpha, \hat{\lambda}_1^\alpha, \hat{\lambda}_2^\alpha, ... \in C$ which violate neither (i) nor (ii) of Definition 1.29.[15] Note that for each $n$, we can select $\vec{\lambda}(n) = (\lambda_1(n), \lambda_3(n), \lambda_5(n), ..., \lambda_{2n-1}(n))$ such that

(xxvi) Neither (i) nor (ii) of Definition 1.29 is violated.

(xxvii) If $\lambda_i(n) = \lambda_j^\alpha$ (i.e. $\pi(\alpha, j) = i$) and $\alpha < \beta$, then $\lambda_j^\alpha = \hat{\lambda}_j^\alpha$.

(xxviii) If $\lambda_i(n) = \lambda_j^\alpha$ (i.e. $\pi(\alpha, j) = i$) and $\alpha > \beta$, then $\lambda_j^\alpha \in C \setminus \omega_{n+1}$, where $n$ is least such that $\omega_{n+1} > \beta$.

Let $\lambda_0(n), \lambda_2(n), \lambda_4(n), ..., \lambda_{2n}(n))$ and $i_0, i_1, ..., i_n$ be such that

$$\left(T; \langle 1, -\rangle; T_0; \langle 0, \bar{x}(i_0)\rangle; x(0), \lambda_0(n); x(1), \lambda_1(n); T_1; \langle 0, \bar{x}(i_1)\rangle; x(2), \lambda_2(n); x(3), \lambda_3(n); ...; T_n; \langle 0, \bar{x}(i_n)\rangle; x(2n), \lambda_{2n}(n)\right)$$

is consistent with $S_+^{\gamma,\beta}$. For even $\alpha \leq \beta$ and sufficiently large $n$ and $\hat{n}$, by using properties of indiscernibles, we get $\lambda_j^\alpha(n) = \lambda_j^\alpha(\hat{n})$ since each $\lambda_j^\alpha(n)$ is required to be played $< \omega_{n+1}$, since $S_+^{\gamma,\beta} \in M$, and by (xxvii) and (xxviii). For even $\alpha \leq \beta$, let $\lambda_j^\alpha$ be the common value of the $\lambda_j^\alpha(n)$ (for $n$ sufficiently large). $x \in D_\alpha$ since the map taking $j$ to the common value of the $\lambda_j^\alpha(n)$'s gives an order-preserving map from $\omega$ under the ordering $<_x^\alpha$ into the ordinals. $x \notin \bigcap_{\alpha<\omega\cdot m} D_\alpha$ (in this case) since otherwise one can write down a play consistent with $S_+^{\gamma,\beta}$ that is a win for player II. By (xxv) and $x \notin \bigcap_{\alpha<\omega\cdot m} D_\alpha$, $x \in D$.

**Lemma 2.14.** If $(\beta+1)\,\#_{\gamma+1}\,(r)$ exists and $S_+^{\gamma,\beta}$ is a w.s. for II, then player II has a w.s. $s_+ \in M^\# = L\left((\beta+1)\,\#_{\gamma+1}\,(r)\right)$ for the game $E_+ = \left(B, \vec{C}\right)^* \left(\vec{A}, D\right)$.

**Proof:** Assume the hypotheses. Let $C$ be a class of indiscernibles for $M$, closed and unbounded beneath every uncountable cardinal.

**Construction of $s_+$:** Let $T$ be as in (xiii), replacing $S^{\gamma,\beta}$ with $S_+^{\gamma,\beta}$, i.e.

(xxix) $T = \{$ positions $t$ in the game $E_+ \mid \forall \tilde{T} \in M \;\; \langle 0, t\rangle \neq S_+^{\gamma,\beta}\left(\tilde{T}\right)\}$.

Then $S_+^{\gamma,\beta}(T) = \langle 1, -\rangle$.

Suppose $(x(0); x(1); x(2); \cdots; x(2n))$ is consistent with $s_+$ and $\forall j \leq 2n+1 \;\bar{x}(j) \in T$. Pick

$$\lambda_0(n), \lambda_2(n), \cdots, \lambda_{2n}(n) \in C$$

so that there exist $\lambda_1(n), \lambda_3(n), \lambda_5(n), ..., \lambda_{2n-1}(n)$ such that

[15] Selecting the $\hat{\lambda}_i^\alpha$'s from $C$ ensures that $s_+$ is well-defined, but serves no additional purpose here when $\alpha < \beta$.

$$q_n = \Big(T; \langle 1, - \rangle ; x(0), \lambda_0(n); x(1), \lambda_1(n), x(2), \lambda_2(n), ..., x(2n), \lambda_{2n}(n)\Big)$$

is consistent with $S_+^{\gamma,\beta}$.[16] Let $s_+ (x(0); x(1); x(2); \cdots; x(2n)) =_{df} x(2n+1)$, where

$$\exists \lambda_{2n+1}(n) \Big( (x(2n+1), \lambda_{2n+1}(n)) = S_+^{\gamma,\beta}(q_n) \Big).$$

Suppose $(x(0); x(1); x(2); \cdots; x(n-1))$ is consistent with $s_+$, $\bar{x}(n) \notin T$, and $\forall j < n$, $\bar{x}(j) \in T$.[17] By the definition of $T$, $\exists \tilde{T} \in M$ such that $\langle 0, \bar{x}(n) \rangle = S_+^{\gamma,\beta}\left(\tilde{T}\right)$. Then $R(\bar{x}(n))$ since $S_+^{\gamma,\beta}$ is a w.s. for II, and $S_{\left(\tilde{T};\langle 0,\bar{x}(n)\rangle\right)} \in M$ is II's w.s. for the game $E_+ (R; T; \langle 0, \bar{x}(n) \rangle) = E(R; T; \langle 0, \bar{x}(n) \rangle) = E(\bar{x}(n))$. Let $s_+(p) =_{df} S_{\left(\tilde{T};\langle 0,\bar{x}(n)\rangle\right)}(p)$ for any position $p$ compatible with $\bar{x}(n)$.

Note that $s_+ \in M^{\#}$.

**Claim.** $s_+$ is a w.s. for player II.

Let $x$ be a play of the game $E_+$ which is consistent with $s_+$. First suppose $\exists^\mu i\ \bar{x}(i) \notin T$, where $T$ is as in (xxix). Then by the definition of $s_+$, $x$ is consistent with II's w.s. $S_{\left(\tilde{T},\langle 0,\bar{x}(i)\rangle\right)}$ for the game $E_+(\bar{x}(i))$ so that $x \notin E_+(\bar{x}(i))$. Since $x$ extends $\bar{x}(i)$, $x \notin E_+$.

Now suppose $\forall i\ \bar{x}(i) \in T$. Then by the definition of $s_+$, there exists a position

$$q_n = (T; \langle 1, - \rangle ; x(0), \lambda_0(n); x(1), \lambda_1(n); ...; x(2n), \lambda_{2n}(n))$$

consistent with $S_+^{\gamma,\beta}$ and such that $\lambda_0(n), \lambda_2(n), \lambda_4(n), \cdots, \lambda_{2n}(n) \in C$. Since each $q_n$ is consistent with II's w.s. $S_+^{\gamma,\beta}$, $\forall i \neg R(\bar{x}(i))$ so that $\forall n \neg \left(\vec{C}\right)(x, n)$ and:

$$x \in E_+ \Leftrightarrow x \in (B)^* \left(\vec{A}, D\right).$$

If $\forall n \neg B(x, n)$, then

$$x \in E_+ \Leftrightarrow x \in D,$$

and using indiscernibles for $M$, integrate II's w.s. $S_+^{\gamma,\beta}$ with respect to the $\lambda_i$'s to get $x \notin D$. (Such integration is analogous to that described in Remark 2.13 with the parity of $\alpha$ and $\beta$ changed throughout Remark 2.13: We show (xxv) for $\beta$ odd, instead of for $\beta$ even as in Remark 2.13, and $x \in \bigcap_{\alpha<\omega\cdot m} D_\alpha$ does not result in a contradiction as at the end of Remark 2.13.) Therefore, if $\forall n \neg B(x, n)$, then $x \notin E_+$.

So assume $\exists n B(x, n)$. By (iii), $x \in \bigcap_{\alpha<\omega\cdot m} D_\alpha$ so that there exists a sequence $\langle \lambda_{2i} \mid i < \omega \rangle$ properly ordered with respect to $x$ and $\langle D_\alpha | \alpha < \omega \cdot m, \alpha \text{ even} \rangle$, and such that each $\lambda_{2i} \in C$. Then $\exists \langle \lambda_{2i+1} | i < \omega \rangle$ such that the play

$$y = (T; \langle 1, - \rangle ; x(0), \lambda_0; x(1), \lambda_1; x(2), \lambda_2; \cdots; x(i), \lambda_i; \cdots)$$

is consistent with II's w.s. $S_+^{\gamma,\beta}$ so that $x \notin B^* \left(\vec{A}\right)$. Since $\exists n B(x, n)$, $x \notin E_+$. ∎

The Borel auxiliary moves $T$ and $\langle \hat{t}, t \rangle$ are not necessary to prove Theorem 2.1. Instead define auxiliary games $\tilde{G}^{\gamma,\beta}$ and $\tilde{G}_+^{\gamma,\beta}$ as follows. In $\tilde{G}^{\gamma,\beta}$ integer moves $x(i)$ are played until a position $(x(0); x(1); x(2); \cdots; x(\ell - 1))$

[16] See Lemma 2.12 for more details on the existence of such $\lambda_i(n)$'s.
[17] $n$ is odd since $T$ is I-imposed (as shown in Lemma 2.9).

is reached such that $R(\bar{x}(\ell))$. If no such position is reached, then

$$\text{I wins } \tilde{G}^{\gamma,\beta} \Leftrightarrow_{df} x \in (B)^* \left(\vec{A}\right).$$

If such a position is reached, then $R^{(n)_0}(\bar{x}(\ell), (n)_1)$ for some $n$, and:

$$\text{I wins } \tilde{G}^{\gamma,\beta} \Leftrightarrow_{df} \text{ player I has a w.s. for the game } \left(\left(B, \vec{C}_{(n)_0}\right)^* \left(\vec{A}, dk\left(\vec{A}_{\omega\cdot(n)_1}\right)\right)\right)(\bar{x}(\ell)).$$

Induction is used to get w.s. for such games $\left(\left(B, \vec{C}_{(n)_0}\right)^* \left(\vec{A}, dk\left(\vec{A}_{\omega\cdot(n)_1}\right)\right)\right)(\bar{x}(\ell))$.

Since the game $G^{\gamma,\beta}$ is clopen and its game tree is definable in $M$, $G^{\gamma,\beta}$ has (see Remark 2.7) a strategy $S^{\gamma,\beta} \in M$ which $M$ believes to be a w.s. $\tilde{G}^{\gamma,\beta}$ is a boldface $(\Gamma)^{**}$ game with code in $M$, and we use a hypothesis that such games have a strategy $S^{\gamma,\beta} \in M$ which $M$ believes to be a w.s. (see (HYP) in §3). Just as one notes $S^{\gamma,\beta}$ is in fact a w.s., we note that $\tilde{S}^{\gamma,\beta}$ is an actual w.s. Using $\tilde{S}^{\gamma,\beta}$, we define a w.s. in $M = L(\beta\#_{\gamma+1}(r))\left[\vec{\#}_\gamma\right]$ for $E$.

$\tilde{G}^{\gamma,\beta}_+$ is defined similarly to $G^{\gamma,\beta}$ except an ordinal auxiliary move $\lambda_i$ is played with each integer move $x(i)$ and the $\lambda_i$'s are required to be properly ordered with respect to $\vec{D}$. $\tilde{G}^{\gamma,\beta}_+$ similarly has a w.s. $\tilde{S}^{\gamma,\beta}_+ \in M = L(\beta\#_{\gamma+1}(r))\left[\vec{\#}_\gamma\right]$. Integrating $\tilde{S}^{\gamma,\beta}_+$ with respect to the $\lambda_i$'s, we obtain a w.s. in $M^\# = L((\beta+1)\#_{\gamma+1}(r))$ for $E_+$.

In [Du90], we prove the determinacy results for $\left(\beta * \Sigma^0_1\right)^*$ and $\left(\beta * \Sigma^0_1\right)^*_+$ using this type of auxiliary game in which: integer moves are played until a position is reached witnessing that one of $\Sigma^0_1$ sets holds, at which point the remainder of the game is of smaller complexity and thus has a w.s. in the appropriate model.

The auxiliary games of the next section will have from 1 to $\omega$ many pairs of Borel auxiliary moves, so our use of $T$ and $\langle \hat{t}, t\rangle$ in this section may be helpful for reading Section 3.