Section 3

In this section, we show the following:

Corollary 3.1.1. Let r be a real and $\gamma < \omega_1^{CK}(r)$.

(i) If $L(r)\left[\vec{\#}^1_\gamma\right] \models$ "$\forall\hat{\gamma} < \gamma \ \#^1_{\hat{\gamma}+1}$ is total", then $\mathrm{Det}\big(\gamma * \Pi^0_1\big)^*(r)$.

(ii) If $\#^1_{\gamma+1}(r)$ exists (i.e. if indiscernibles for $L(r)\left[\vec{\#}^1_\gamma\right]$ exist), then $\mathrm{Det}\big(\gamma * \Pi^0_1\big)^*_+(r)$.

Our proof also provides determinacy results for other models $L(r)[\vec{\mathcal{S}}^\gamma_1(\vec{\#})]$ which are closed under $\vec{\mathcal{S}}^\gamma_1(\vec{\#})$ and for the existence of indiscernibles for such models.

Theorem 3.1. Fix r a real and $\vec{\#} = \vec{\mathcal{S}}^{\gamma(n)}_{k(n)}\vec{\mathcal{S}}^{\gamma(n-1)}_{k(n-1)}\cdots\vec{\mathcal{S}}^{\gamma(1)}_{k(1)}\vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset)$, where each $\gamma(i) < \omega_1^{CK}(r)$ and $\omega_1^{CK}(r) > k(1) > k(2) > \cdots > k(n)$. Let $\Gamma = \Gamma_r(\vec{\#})$ (see Definition 1.20 for the definition of $\Gamma_r(\vec{\#})$). For $\gamma < \omega_1^{CK}(r)$, let $\vec{\#}_\gamma = \vec{\mathcal{S}}^\gamma_1(\vec{\#})$ and $\#_{\gamma+1} = \mathcal{S}^\gamma_1(\vec{\#}_\gamma)$. Fix $\gamma < \omega_1^{CK}(r)$.

(i) Let $\gamma \geq 1$ in the case $n = 0$ (i.e. in the case $\vec{\#} = \vec{\#}^{\gamma(0)\infty}$). If $L(r)\left[\vec{\#}_\gamma\right] \models$ "$\forall\beta < \gamma \ \#_{\beta+1}$ is total", then every $\big(\Gamma, \gamma * \Pi^0_1\big)^*(r)$ game has a w.s. in $L(r)\left[\vec{\#}_\gamma\right]$. Hence, if γ is a successor ordinal and $L(r)\left[\vec{\#}_\gamma\right] \models$ "$\#_\gamma$ is total", then every $\big(\Gamma, \gamma * \Pi^0_1\big)^*(r)$ game has a w.s. in $L(r)\left[\vec{\#}_\gamma\right]$.

(ii) If $\#_{\gamma+1}(r)$ exists (i.e. indiscernibles for $L(r)\left[\vec{\#}_\gamma\right]$ exist), then every $\big(\Gamma, \gamma * \Pi^0_1\big)^*_+(r)$ game has a w.s. in $L(\#_{\gamma+1}(r))$.

Corollary 3.1.1 follows from Theorem 3.1 by taking $\vec{\#} = \emptyset$. Also of special interest are the cases where $\vec{\#} = \#^k$ (see Corollary 3.1.2) and $\vec{\#} = \vec{\#}^{k\infty}$ (see Corollary 3.1.3). These two cases are the analogues of Corollary 3.1.1 for the models $L(r)\left[\vec{\#}^{k,1}_\gamma\right]$ and $L(r)\left[\vec{\#}^{k\infty,1}_\gamma\right]$.

Corollary 3.1.2. Let r be a real and $k, \gamma < \omega_1^{CK}(r)$.

(i) If $L(r)\left[\vec{\#}^{k,1}_\gamma\right] \models$ "$\forall\hat{\gamma} < \gamma \ \#^{k,1}_{\hat{\gamma}+1}$ is total", then $\mathrm{Det}\big(\Pi^0_k, \gamma * \Pi^0_1\big)^*(r)$.

(ii) If $\#^{k,1}_{\gamma+1}(r)$ exists (i.e. indiscernibles for $L(r)\left[\vec{\#}^{k,1}_\gamma\right]$ exist), then $\mathrm{Det}\big(\Pi^0_k, \gamma * \Pi^0_1\big)^*_+(r)$.

Corollary 3.1.3. Let r be a real and $k, \gamma < \omega_1^{CK}(r)$.

(i) If $\gamma \neq 0$ and $L(r)\left[\vec{\#}^{k\infty,1}_\gamma\right] \models$ "$\forall\hat{\gamma} < \gamma \ \#^{k\infty,1}_{\hat{\gamma}+1}$ is total", then $\mathrm{Det}\big(k * \Pi^1_1, \gamma * \Pi^0_1\big)^*(r)$.

(ii) If $\#^{k\infty,1}_{\gamma+1}(r)$ exists, then $\mathrm{Det}\big(k * \Pi^1_1, \gamma * \Pi^0_1\big)^*_+(r)$.

It should be clear that analogous to Theorem 3.1, we could also state in Corollaries 3.1.1, 3.1.2, and 3.1.3 the models which contain the winning strategies, and include the usual "reduction" to the hypothesis of part (i) of each corollary for the case in which γ is a successor.

Theorem 3.1 and Corollaries 3.1.1, 3.1.2, 3.1.3 hold even when r is of a higher type than a real, and boldface analogues of these results also hold. Our interest here is to prove these results for the case in which

$r = \emptyset$. The proof in this section shows Corollary 3.1.1 by induction on γ. However, the more general of the above listed results follow by instead showing an appropriate boldface analogue–see Remark 1.22–(which is similarly established by induction). We instead prove the above results assuming an appropriate boldface analogue of the case of (i) of these results in which $\gamma = 0$. This analogue, named (HYP), is given below. Other than assuming (HYP), the proof has the appearance of being an induction on γ, with the "induction hypothesis" including the relativization to reals. This needed relativization is the reason we stated relativized versions of our results.

For the remainder of this section, let us fix r, $\vec{\#}$, and Γ as in the hypothesis of Theorem 3.1. (Shortly we shall similarly fix γ.) We need to assume some analogue, called (HYP) below, of (i) for $\gamma = 0$ but on game trees with which include noninteger moves. We may either resort to stating (HYP) for boldface $(\mathbf{\Gamma})^{**}$ with appropriate codes or resort to defining lightface $(\Gamma)^{*}$ for more general game trees. A natural choice is the following:

(HYP)$'$ If z is a real and $L(z)[\vec{\#}] \models$ "every $\#$ in $\vec{\#}$ is total," then every boldface $(\mathbf{\Gamma})^{*}_{+}$ game with code in $L(z)[\vec{\#}]$ has a w.s. in $L(z)[\vec{\#}]$.

(HYP)$'$ causes problems because of nonabsoluteness of winning strategies, specifically because of the possible lack of winning strategies for the game G_n of Definition 3.3 (see the last sentence of Remark 3.4). Instead we state (HYP) in terms of the model $L(r)[\vec{\#}_\gamma]$ which will contain the winning strategies for the auxiliary games of the section.

(HYP) If G is a boldface $(\mathbf{\Gamma})^{**}$ game with a code (r, T, S^1, S^2) such that $S^1, S^2 \in L(r)$ and $T \in L(r)[\vec{\#}_\gamma]$, then G has a w.s. in $L(r)[\vec{\#}_\gamma]$.

The more natural but stronger version of (HYP) obtained by allowing S^1 and S^2 to range over $L(r)[\vec{\#}_\gamma]$ also holds.

Now we fix $\gamma \in [1, \omega_1^{CK}(r))$. Our "induction hypothesis" for proving (i) and (ii) of Theorem 3.1 is the following analogue of (ii):

(IH) If z is a real, $\beta < \gamma$, and $\#_{\beta+1}(z)$ exists, then every $\left(\Gamma, \beta * \Pi^0_1\right)^*_+(z)$ game has a w.s. in $L\left(\#_{\beta+1}(z)\right)$.

We emphasize that (HYP) and (IH) are being used under the condition that $L(r)[\vec{\#}_\gamma] \models$ "each $\#$ in $\vec{\#}$ is total." The proof of Corollary 3.1.1 does not use (HYP), and in this case $\Gamma = \emptyset$ and (HYP) says something obvious.

We have tried to set up the notation so that this section may be read as a proof to any of Corollaries

3.1.1, 3.1.2, and 3.1.3. For Corollary 3.1.1, set $\vec{\#} = \emptyset$ so that $\Gamma = \Gamma(\vec{\#}) = \emptyset$. For Corollary 3.1.2, set $\vec{\#} = \#^k$ so that $\Gamma = \Gamma(\vec{\#}) = \Pi^0_k$. For Corollary 3.1.3, set $\vec{\#} = \#^{k\infty}$ so that $(\Gamma) = (\Gamma(\vec{\#})) = (k * \Pi^1_1)$. Corollary 3.1.1 is shown for $\gamma = 1$ in [Du92a]. Corollary 3.1.2 is shown for $\gamma = 0$ in [Du95]. (ii) of Corollary 3.1.3 for $\gamma = 0$ is shown in [Du7], but (i) without the restriction that $\gamma \neq 0$ is false.[18] These may serve as the base step for proving the Corollaries.

Fix $C \in \Gamma(r)$, $\vec{B} = \langle B_\alpha | \alpha < \gamma \rangle$, $\vec{A}$, and $D \in \left(\omega \cdot m - \Pi^1_1\right)(r)$, for some $m < \omega$, which witness $\left(C, \vec{B}\right)^* \left(\vec{A}, D\right) \in \left(\Gamma, \gamma * \Pi^0_1\right)^*_+ (r)$ and such that:

(iii) If $B_{\hat{\gamma}}(x,n)$, then

$$x \in (C, \vec{B})^*(\vec{A}, D) \Leftrightarrow x \in (C, \vec{B})^*(\vec{A}) \Leftrightarrow x \in (C, \vec{B}_{\hat{\gamma}})^*(\vec{A}, dk(\vec{A}_{\omega \cdot n})).$$

If $B^\mu_{\hat{\gamma}}(x,n)$, then the conclusion of (iii) follows directly without the Normal Form Theorem. The proofs of this section can be simplified for the case in which γ is finite, and (iii) is not used in the simplified proof (See Remark 3.7).

For $i < \gamma$, let $R^i \in \Delta^0_1(r)$ be such that:

(iv) $B_i(x,n) \Leftrightarrow \forall \ell \neg R^i(\bar{x}(\ell), n)$

(v) If $R^i(\bar{x}(\ell), n)$ and $\forall j < \ell \neg R^i(\bar{x}(j), n)$, then ℓ is odd.

Let $n \longmapsto ((n)_0, (n)_1)$ be a recursive bijection from ω onto $\gamma \times \omega$. Let

(vi) $R_n(q) \Leftrightarrow R^{(n)_0}(q, (n)_1)$.

Let $\vec{D} = \langle D_\alpha | \alpha < \omega \cdot m \rangle$ witness that D is $\omega \cdot m - \Pi^1_1$ and (by the Normal Form Theorem) satisfy

(vii) each $D_\alpha \supseteq \{x | \exists n C(x,n)\}$.

We prove Theorem 3.1 by showing (under the appropriate hypotheses) that the games $\left(C, \vec{B}\right)^* \left(\vec{A}\right)$ and $\left(C, \vec{B}\right)^* \left(\vec{A}, D\right)$ have winning strategies (in the corresponding models). In Definition 3.3, we define open auxiliary games G^γ and G^γ_+, respectively corresponding to the games $\left(\vec{B}\right)^* \left(\vec{A}\right)$ and $\left(\vec{B}\right)^* \left(\vec{A}, D\right)$.[19] We complete the proof to Theorem 3.1 by showing in Lemmas 3.8–3.11 that for $\lambda = I, II$, if player λ has a w.s. for G^γ [respectively G^γ_+], then he has one in $L(r)\left[\vec{\#}_\gamma\right]$ [respectively $L(\#_{\gamma+1}(r))$] for the game $\left(C, \vec{B}\right)^* \left(\vec{A}\right)$ [respectively $\left(C, \vec{B}\right)^* \left(\vec{A}, D\right)$]. It will be convenient to first define $A(R;T;p)$, which is similar to $A(S;T;\langle 1, - \rangle) = A(S;T)$ (see Definition 1.26).

Definition 3.2. Suppose the pair T and $\langle 1, - \rangle$ of Borel auxiliary moves are determined by $R \subseteq \omega$. Let A be a set of reals and let p be a position in the game A. We define the game $A(R;T;p)$ as follows: If

[18] See the sentence preceding Proposition 2.5.

[19] Actually G^γ [respectively G^γ_+] depends on C, $\langle R_n | n < \omega \rangle$, $\vec{A}$ [respectively, and D].

$\forall i\, \bar{x}(i) \in T \setminus R$ and x extends p, then

$$x \in A(R;T;p) \Leftrightarrow x \in A.$$

If for some least i we have

$$R(\bar{x}(i)) \text{ or } \bar{x}(i) \notin T \text{ or } \bar{x}(i) \text{ is incompatible with } p,$$

then

(viii) $x \in A(R;T;p) \Leftrightarrow$ either i is even or (i is odd $\wedge$ $\bar{x}(i) \in T$ $\wedge$ $\bar{x}(i)$ and p are compatible). △

In the game $A(R;T;p)$, the moves are to be consistent with the position p, player I "needs" each position to be in T, and player II "needs" no postion to be in R. If $R(\bar{x}(i))$ or $\bar{x}(i) \notin T$, then i is odd (i.e. the last move of $\bar{x}(i)$ is player I's); so if $x \in A(R;T;p)$ because the i in (viii) is even, then II lost by playing a move incompatible with p. If instead $x \in A(R;T;p)$ and the i in (viii) is odd, then II lost because player I played a move resulting in a position in $T \cap R$ and compatible with p.

Definition 3.3. Recall r, C, $\vec{B}$, $\langle R_n | n < \omega \rangle$, $\vec{A}$, and $\vec{D}$, all fixed earlier this section. Let

$$A_n = \left(C, \vec{B}_{(n)_0}\right)^* \left(\vec{A}, dk\left(\vec{A}_{\omega\cdot(n)_1}\right)\right).$$

We simultaneously define the open auxiliary games G^γ and G^γ_+, since they are very similar.

We rely on suggestive notation in our description below of G^γ and G^γ_+, first listing typical plays of G^γ and G^γ_+ and then describing the moves of these games. Assuming the players meet all the necessary conditions (specified below), typical plays of G^γ and G^γ_+ are of the following form:

(1) $G^\gamma \begin{array}{l|cccccccccccc} \text{I} & T_0 & & x(0) & & T_1 & & x(2) & & \cdots & T_n & & x(2n) & & \cdots \\ \text{II} & & \langle 0,t_0\rangle & & x(1) & & \langle 0,t_1\rangle & & x(3) & & & \langle 0,t_n\rangle & & x(2n+1) & \end{array}$

(2) $G^\gamma_+ \begin{array}{l|cccccccccccc} \text{I} & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & & \cdots & T_n & & x(2n),\lambda_{2n} & & \cdots \\ \text{II} & & \langle 0,t_0\rangle & & x(1),\lambda_1 & & \langle 0,t_1\rangle & & x(3),\lambda_3 & & & \langle 0,t_n\rangle & & x(2n+1),\lambda_{2n+1} & \end{array}$

(3) $G^\gamma \begin{array}{l|cccccccccccc} \text{I} & T_0 & & x(0) & & T_1 & & x(2) & & \cdots & T_{n-1} & & x(2n-2) & & T_n & \\ \text{II} & & \langle 0,t_0\rangle & & x(1) & & \langle 0,t_1\rangle & & x(3) & & & \langle 0,t_{n-1}\rangle & & x(2n-1) & & \langle 1,-\rangle \end{array}.$

(4) $G^\gamma_+ \begin{array}{l|cccccccccccc} \text{I} & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & & \cdots & T_{n-1} & & x(2n-2),\lambda_{2n-2} & & T_n & \\ \text{II} & & \langle 0,t_0\rangle & & x(1),\lambda_1 & & \langle 0,t_1\rangle & & x(3),\lambda_3 & & & \langle 0,t_{n-1}\rangle & & x(2n-1),\lambda_{2n-1} & & \langle 1,-\rangle \end{array}.$

In G^γ and G^γ_+, whenever $\forall i < n\ \hat{t}_i = 0$, Borel auxiliary moves T_n, $\langle \hat{t}_n, t_n \rangle$ are played during the $(2n+1)^{\text{st}}$ inning and integer moves $x(2n)$ and $x(2n+1)$ are played during the $(2n+2)^{\text{nd}}$ inning. In G^γ_+ (but not in G^γ), an ordinal auxiliary move λ_i is played with the integer move $x(i)$. Once $\hat{t}_n = 1$ is played, there is no need to play any more moves - see (3) and (4) above.

(∗) In both G^γ and G^γ_+, player I may only play $T_n \in L(r)\left[\vec{\#}_\gamma\right]$ and each pair of Borel auxiliary moves T_n and $\langle \hat{t}_n, t_n \rangle$ are determined by the $\Sigma^0_1(r)$ set R_n.

(∗∗) In G^γ_+, the ordinal auxiliary moves λ_i's must be properly ordered with respect to $\vec{D}$ using $\langle \omega_{i+1} | i < m\rangle$.

The first player to violate either (∗) or (∗∗) loses. Note that we have implicitly defined the game trees T^γ and T^γ_+ for G^γ and G^γ_+. Let

$$i_{\max}(2n) = \max\{2n, \mathit{length}(t_i) | i < n\}.$$

If neither player violates (∗) [and (∗∗)], then:

I wins G^γ [respectively, G^γ_+] iff

$$\exists n \left(\hat{t}_n = 1 \text{ and player I has a w.s. for the game } G_n =_{df} A_n\left(R_n; T_n; \bar{x}\left(i_{\max}(2n)\right)\right)\right)$$ [20]

or

$$\left(\forall n\ \hat{t}_n = 0 \text{ and } x \in (C)^*(\vec{A})\right).$$

Let A^γ [respectively A^γ_+] be the payoff set for G^γ [respectively G^γ_+]. △

If q is a position of G^γ or G^γ_+, let

$$i_{\max}(q) = \max\{i+1, \mathit{length}(t_i) | x(i);\ x(i), \lambda_i;\text{ or } \langle \hat{t}_i, t_i\rangle \text{ is a move of } q\}.$$

When $\hat{t}_i = 1$, $\mathit{length}(t_i) = 0$. If q is a position whose last move includes $x(2n-1)$, then $i_{\max}(q) =_{df} i_{\max}(2n)$.

Remark 3.4. Assume $L(r)\left[\vec{\#}_\gamma\right] \models$ "$\forall \beta < \gamma\ \#_{\beta+1}$ is total". If we reach a legal position p in either G^γ or G^γ_+ whose last move is $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$ and if x is a legal play of $G_n = A_n\left(R_n; T_n; \bar{x}\left(i_{\max}(2n)\right)\right)$, then:

$$x \text{ is a win for I iff } x \in A_n.$$

Since $A_n \in \left(\Gamma, (n)_0 * \Pi^0_1\right)^*_+ (r, T_n)$ and $(n)_0 < \gamma$, by induction hypothesis the game G_n has a w.s. $s_p \in L\left(\#_{(n)_0}(r, T_n)\right) \subseteq L(r)\left[\vec{\#}_\gamma\right]$. Also by (∗) of Definition 3.3, each $T_n \in L(r)\left[\vec{\#}_\gamma\right]$, and $T^\gamma, T^\gamma_+ \in L(r)\left[\vec{\#}_\gamma\right]$. Consequently, for $s^\gamma, s^\gamma_+ \in L(r)[\vec{\#}_\gamma]$:

$$L(r)[\vec{\#}_\gamma] \models \text{“}s^\gamma \text{ is a w.s. for the game } G^\gamma\text{”} \leftrightarrow s^\gamma \text{ is a w.s. for the game } G^\gamma.$$

$$L(r)[\vec{\#}_\gamma] \models \text{“}s^\gamma_+ \text{ is a w.s. for the game } G^\gamma_+\text{”} \leftrightarrow s^\gamma_+ \text{ is a w.s. for the game } G^\gamma_+.$$

(Note that this may be false for models which do not contain winning strategies for the games G_n.)

The games G^γ and G^γ_+ are boldface $(\mathbf{\Gamma})^{**}$ (open in the case $C = \emptyset$) respectively with codes (r, T^γ, S_1, S_2) and $(r, T^\gamma_+, S_1, S_2)$ such that $S_1, S_2 \in L(r)$. Therefore, by (HYP), G^γ and G^γ_+ respectively have winning strategies s^γ and s^γ_+ in $L(r)\left[\vec{\#}_\gamma\right]$. (In fact G^γ and G^γ_+ respectively have strategies in $L(T^\gamma, \#_1(r))$ and $L(T^\gamma_+, \#_1(r))$ which these models believe are winning strategies for their respective games, but these may not be winning strategies in the real world since $L(T^\gamma, \#_1(r))$ and $L(T^\gamma_+, \#_1(r))$ may fail to contain winning strategies for the games G_n.)

[20] Recall A_n is defined at the beginning of Definition 3.3.

Remark 3.5. Consider the case of Corollary 3.1.1 in which $\vec{\#} = \emptyset$. $C = \emptyset$ so that the second disjunct of the winning conditions for G^γ and G^γ_+ cannot hold, and player I wins G^γ only if $\exists n\, \hat{t}_n = 1$, whereas player I may win G^γ_+ for a play in which no $\hat{t}_i = 1$ is played, if for instance II cannot play some ordinal auxiliary move λ_{2n}. When $\vec{\#} = \emptyset$, G^γ and G^γ_+ are open games, and so the Gale-Stewart Theorem in $L(r)[\vec{\#}_\gamma]$ provides us with winning strategies s^γ and s^γ_+.

Lemma 3.6. Recall C, $\vec{B}, \vec{A}$, and γ are fixed. For x a real, let $N(x)$ be the least N which satisfies:

$$N = \omega \text{ or } B_{(N)_0}(x, (N)_1).$$

(a) $\forall n < N(x) \exists i_n\, R_n(\bar{x}(i_n))$.

(b) If $N(x) < \omega$, then $\forall j\, \bar{x}(j) \notin R_{N(x)}$.

(c) If $N(x) < \omega$, then: $x \in A_{N(x)} \Leftrightarrow x \in \left(C, \vec{B}\right)^* \left(\vec{A}\right)$.

Proof: Parts 3.6(a) and 3.6(b) follow from the definitions of N and R_N. 3.6(c) follows immediately from (iii). If $N = N(x) < \omega$, then $B_{(N)_0}(x, (N)_1)$ so that by (iii),

$$x \in A_N = (C, \vec{B}_{(N)_0})^*(\vec{A}, dk(\vec{A}_{\omega\cdot(N)_1})) \Leftrightarrow x \in (C, \vec{B})^*(\vec{A}).$$

∎(Lemma 3.6)