

### Section 3—continued

**Remark 3.7.** By the Normal Form Theorem 1.19, we may assume wlog that

$$(*) \quad B_\alpha \subseteq B_{\alpha+1} \text{ for } \gamma_{limit} \leq \alpha < \alpha + 1 < \gamma,$$

where  $\gamma_{limit}$  is the largest limit ordinal  $< \gamma$  ( $\gamma_{limit} = 0$  when  $\gamma < \omega$ ). In the proofs of this section, we consider (in our auxiliary games  $G^\gamma$  and  $G_+^\gamma$ ) a pair of Borel auxiliary moves corresponding to each element of  $\gamma \times \omega$  (more precisely, determined by each  $R_n = \{q|R^{(n)0}(q, (n)_1)\}$ ). For the case when  $\gamma$  is a successor, we can slightly simplify our presentation and carry out the proofs with  $R_n$  and  $N(x)$  being respectively replaced by  $\tilde{R}_n = \{q|R^\gamma(q, n)\}$  and  $\tilde{N}(x) = \mu N(B_\gamma(x, N) \text{ or } N = \omega)$ , assuming (\*). Then each pair  $T_n, \langle \hat{t}_n, t_n \rangle$  of Borel auxiliary moves are instead determined by the  $\Sigma_1^0(r)$  set  $\tilde{R}_n = \{q|R^\gamma(q, n)\}$ , and  $G_n$  is replaced by  $\tilde{G}_n = A_n(\tilde{R}_n; T_n; \bar{x}(i_{max}(2n)))$ . The second disjunct of the winning conditions for  $G^\gamma$  and  $G_+^\gamma$  also is slightly changed (specifically  $(C, \vec{B}_{\gamma_{limit}})$  replaces  $(C)$ ):

I wins  $\tilde{G}^\gamma$  [respectively,  $\tilde{G}_+^\gamma$ ] iff either

$$\exists n \left( \hat{t}_n = 1 \text{ and player I has a w.s. for the game } G_n =_{df} A_n(R_n; T_n; \bar{x}(i_{max}(2n))) \right)$$

or

$$(\forall n \hat{t}_n = 0 \text{ and } x \in (C, \vec{B}_{\gamma_{limit}})^*(\vec{A})).$$

The games  $\tilde{G}^\gamma$  and  $\tilde{G}_+^\gamma$  are boldface  $(\Gamma, \gamma_{limit} * \Pi_1^0)^{**}$ , and so we need (HYP) for boldface  $(\Gamma, \gamma_{limit} * \Pi_1^0)^{**}$  games instead of just  $(\Gamma)^{**}$  games.

For  $\gamma$  a limit, we need (iii) because when  $N = N(x) < \omega$ ,  $B_{(N)_0}^\mu(x, (N)_1)$  may fail (even though  $B_{(N)_0}(x, (N)_1)$  holds). However, if  $\tilde{N} = \tilde{N}(x) < \omega$ , then  $B_\gamma^\mu(x, \tilde{N})$  and (iii) is not required. Recall from Remark 3.4 that when we reach in our auxiliary games a legal position  $p$  whose last move is  $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$ , the game  $G_n$  has a w.s.  $s_p \in L(\#_{(n)_0}(r, T_n)) \subseteq L(r)[\vec{\#}_\gamma]$ . For such a position  $p$ ,  $\tilde{G}_n$  has a w.s.  $s_p \in L(\#_\gamma(r, T_n)) \subseteq L(r)[\vec{\#}_\gamma]$  since  $A_n \in (\Gamma, (\gamma - 1) * \Pi_1^0)_+^*(r, T_n)$ .

**Lemma 3.8.** If  $L(r)[\vec{\#}_\gamma] \models \text{“}\forall \beta < \gamma \#_{\beta+1} \text{ is total”}$  and player I has a w.s. for  $G^\gamma$ , then I has a w.s.  $s \in L(r)[\vec{\#}_\gamma]$  for the game  $(C, \vec{B})^*(\vec{A})$ .

**Proof.** Assume I has a w.s. for  $G^\gamma$ . Then  $s^\gamma \in L(r)[\vec{\#}_\gamma]$  of Remark 3.4 is a w.s. for I.

**Construction of  $s$ :** Let  $T_0 = s^\gamma(\langle \rangle)$  and  $q_0 = (T_0; \langle 1, - \rangle)$ . Recall  $s_{q_0}$  (alias  $s_p$  in Remark 3.4 for  $p = q_0$ ).  $s_{q_0} \in L(\#_{(n)_0}(r, T_0))$  is a w.s. for I in the game  $A_0(R_0; T_0; \emptyset)$ . Set  $s(p) = s_{q_0}(p)$  for any position

$$p = (x(0); x(1); x(2); \dots, x(2i-1)) \text{ such that } \forall j < i \neg R_0(\bar{x}(j)).$$

Suppose we reach a position  $\bar{x}(i_0)$  such that  $R_0^\mu(\bar{x}(i_0))$ . (Recall that for  $R \subseteq \omega$ ,  $R^\mu(\bar{x}(i)) \Leftrightarrow_{df} R(\bar{x}(i)) \wedge \forall j < i \neg R(\bar{x}(j))$ .) By (v),  $i_0$  is odd. Define  $x(0)$  and  $T_1$  so that  $q_1 =_{df} (T_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); T_1; \langle 1, - \rangle)$  is consistent with  $s^\gamma$ . Set  $s(p) = s_{q_1}(p)$  for any position  $p = (x(0); x(1); x(2); \dots; x(2i-1))$  consistent with  $q_1$  and such that  $\forall j < 2i \neg R_1(\bar{x}(j))$ . If we reach a position  $\bar{x}(i_1)$  such that  $R_1^\mu(\bar{x}(i_1))$ , we continue as

above.

In general, suppose we reach a position  $(x(0); x(1); x(2); \dots, x(i_{\max}(2n) - 1))$  consistent with  $s$ , where  $i_{\max}(2n) = \max\{2n, i_j | j = 0, 1, 2, \dots, n-1\}$  and  $\forall j < n R_j^\mu(\bar{x}(i_j))$ . Let  $x(0), x(2), x(4), \dots, x(2n-2)$  and  $T_0, T_1, T_2, \dots, T_n$  be such that

$$q_n = (T_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); T_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots \\ \dots, T_{n-1}; \langle 0, \bar{x}(i_{n-1}) \rangle; x(2n-2); x(2n-1); T_n; \langle 1, - \rangle)$$

is consistent with  $s^\gamma$ .  $s_{q_n} \in L(\#_{(n)_0}(r, T_n))$  is a w.s. for I in the game  $G_n = A_n(R_n; T_n; \bar{x}(i_{\max}(2n)))$ . Set  $s(p) = s_{q_n}(p)$  for any position  $p = (x(0); x(1); x(2); \dots, x(2i-1))$  consistent with  $\bar{x}(i_{\max}(2n))$  and such that  $\forall j < 2i - R_n(\bar{x}(j))$ .

Note that  $s \in L(r) \left[ \vec{\#}_\gamma \right]$ .

**Claim.** The strategy  $s$  of player I is a w.s. for the game  $(C, \vec{B})^* (\vec{A})$ .

Let  $x$  be a play of the game  $(C, \vec{B})^* (\vec{A})$  consistent with  $s$ . Let  $N = N(x)$ , where  $N(x)$  is defined in Lemma 3.6, i.e. let  $N$  be least such that:

$$N = \omega \text{ or } B_{(N)_0}^\mu(x, (N)_1).$$

By Lemma 3.6(a) for each  $j < N$ , there is a least  $i_j$  such that  $R_j^\mu(\bar{x}(i_j))$ . By the definition of  $s$ , for each finite  $k \leq N$ , there exist  $T_0, T_1, T_2, \dots, T_k$  such that the position

$$q_k =_{df} (T_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); T_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots \\ \dots; T_{k-1}; \langle 0, \bar{x}(i_{k-1}) \rangle; x(2k-2); x(2k-1); T_k; \langle 1, - \rangle)$$

is consistent with  $s^\gamma$ .

**Case 1.**  $N = \omega$  (i.e.  $\forall \hat{\gamma} < \gamma \forall n \neg B_{\hat{\gamma}}(x, n)$ ). Then since each  $q_k$  is consistent with  $s^\gamma$ ,

$$y =_{df} (T_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); T_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots; T_n; \langle 0, \bar{x}(i_n) \rangle; x(2n); x(2n+1); \dots)$$

is consistent with player I's w.s.  $s^\gamma$ . Hence  $y \in A^\gamma$  (the payoff set for  $G^\gamma$ ) and  $\forall n \hat{t}_n = \emptyset$  so that  $x \in (C)^*(\vec{A}) \subseteq (C, \vec{B})^*(\vec{A})$ . (Note that  $N$  must be  $< \omega$  in the case  $\vec{\#} = \emptyset$  since when  $\vec{\#} = \emptyset$ ,  $C = \emptyset$  and the win  $y$  for player I is a play of  $G^\gamma$  in which player II made only legal moves and so is a win for II.)

**Case 2.**  $N < \omega$ .

By the definition of  $s$  and since  $N < \omega$ ,  $s(p) = s_{q_N}(p)$  for any position  $p = (x(0); x(1); x(2); \dots; x(i-1))$  compatible with  $\bar{x}(i_{\max}(2N))$  and such that  $\forall j \leq i - R_N(\bar{x}(j))$ . By Lemma 3.6(b),  $\forall j \neg R_N(\bar{x}(j))$ . Therefore,  $x$  is a play consistent with  $s_{q_N}$  so that  $x \in A_N(R_N; T_N; \bar{x}(i_{\max}(2N)))$  and  $x \in A_N$ . By Lemma 3.6(c),  $x \in (C, \vec{B})^* (\vec{A})$ . ■(Lemma 3.8)

Lemmas 3.8 and 3.10 prove part (i) of Theorem 3.1. We next construct (in the proof of Lemma 3.9) a w.s. of player I for the game  $(C, \vec{B})^* (\vec{A}, D)$  since the construction is almost identical to that in Lemma 3.8.

**Lemma 3.9.** Assume  $\#_{\gamma+1}(r)$  exists. Player I has a w.s.  $s \in L(\#_{\gamma+1}(r))$  for the game  $(C, \vec{B})^* (\vec{A}, D)$  if he has one for  $G_+^\gamma$ .

**Proof:** Assume  $C$  is a class of indiscernibles for  $L(r) \left[ \vec{\#}_\gamma \right]$  which is closed and unbounded beneath every uncountable cardinal. Assume I has a w.s. for  $G_+^\gamma$ . Then  $s_+^\gamma \in L(r) \left[ \vec{\#}_\gamma \right]$  of Remark 3.4 is a w.s. for I. We define  $s$  from  $s_+^\gamma$  similarly to how we defined a w.s. for  $(C, \vec{B})^* (\vec{A})$  from  $s^\gamma$  in Lemma 3.8.

**Construction of  $s$ :** Let  $\hat{q}_0 = \emptyset$  and suppose

$$\hat{q}_n = (T_0; \langle 0, \bar{x}(i_0) \rangle; x(0), \lambda_0(n-1); x(1), \lambda_1(n-1); T_1; \langle 0, \bar{x}(i_1) \rangle; x(2), \lambda_2(n-1); x(3), \lambda_3(n-1); \cdots; \\ T_{n-1}; \langle 0, \bar{x}(i_{n-1}) \rangle; x(2n-2); \lambda_{2n-2}(n-1); x(2n-1), \lambda_{2n-1}(n-1))$$

is consistent with  $s_+^\gamma$ , where  $\lambda_1(n-1), \lambda_3(n-1), \cdots, \lambda_{2n-1}(n-1) \in C$ .

Let  $T_n = s_+^\gamma(\hat{q}_n)$  and  $q_n = \hat{q}_n \cap (T_n, \langle 1, - \rangle)$ .  $s_{q_n} \in L(\#_{(n)_0}(r, T_n))$  is a w.s. for I in the game  $A_n(R_n; T_n; \bar{x}(i_{\max}(2n)))$ . (Recall  $i_{\max}(2n) = \max\{2n, i_j | j < n\}$ .) Set  $s(p) = s_{q_n}(p)$  for any position  $p = (x(0), x(1), \cdots, x(2i-1))$  compatible with  $\bar{x}(i_{\max}(2n))$  and such that  $\forall j < 2i \bar{x}(j) \notin R_n$ . Suppose we reach a position  $\bar{x}(i_n)$  such that  $R_n^\mu(\bar{x}(i_n))$ . By (v),  $i_n$  is odd. Let  $(x(2n), \lambda_{2n}) = s_+^\gamma(q_n)$ . Fix  $x(2n+1)$ . Note that we may not be able to find  $\lambda_{2n+1}$  such that  $q_n \cap (x(2n), \lambda_{2n}; x(2n+1), \lambda_{2n+1})$  is a legal position in  $G_+^\gamma$ . Instead pick  $\lambda_1(n), \lambda_3(n), \lambda_5(n), \cdots, \lambda_{2n+1}(n) \in C$  which are properly ordered with respect to  $\bar{x}(i_{\max}(2n+2))$ . Then  $\exists \lambda_0(n), \lambda_2(n), \cdots, \lambda_{2n}(n)$  such that

$$\hat{q}_{n+1} = (T_0; \langle 0, \bar{x}(i_0) \rangle; x(0), \lambda_0(n); x(1), \lambda_1(n); \cdots; T_n, \langle 0, \bar{x}(i_n) \rangle; x(2n), \lambda_{2n}(n); x(2n+1), \lambda_{2n+1}(n))$$

is consistent with  $s_+^\gamma$ .

Note that  $s \in L(\#_{\gamma+1}(r))$ .

**Claim.** The strategy  $s \in L(\#_{\gamma+1}(r))$  of player I is a w.s. for the game  $(C, \vec{B})^* (\vec{A}, D)$ .

Let  $x$  be a play of the game  $(C, \vec{B})^* (\vec{A}, D)$  consistent with  $s$ . Let  $N =_{df} N(x)$ , defined in Lemma 3.6.

**Case 1.**  $N < \omega$ , i.e.  $\exists n \left( \vec{B} \right) (x, n)$ . Then, arguing as in Lemma 3.8, there is a legal position  $q_N$  of  $G_+^\gamma$  with last move  $\langle \hat{t}_N, t_N \rangle = \langle 1, - \rangle$  such that  $q_N$  is consistent with  $s_+^\gamma$  and  $x$  is a w.s. for I in the game  $A_N(R_N; T_N; q_N)$ . Hence  $x \in A_N(R_N; T_N; q_N)$ . Since  $x$  is compatible with  $\bar{x}(i_{\max}(q_N))$ ,  $x \in A_N$  or  $\exists j \bar{x}(j) \notin R_N$ . By Lemma 3.6(b),  $\forall j \bar{x}(j) \in R_N$  so that by Lemma 3.6(c),  $x \in (C, \vec{B})^* (\vec{A})$ .

**Case 2.**  $N = \omega$ , i.e.  $\forall n \neg \left( \vec{B} \right) (x, n)$ . Then for each  $n$ , there is a least  $i_n$  such that  $R_n(\bar{x}(i_n))$ .

First suppose  $\exists n C(x, n)$ . Then by (vii)  $x \in D_\alpha$  for each  $\alpha$  so that we may select  $\langle \lambda_{2i+1}^* | i < \omega \rangle$  from  $C$  for which some play

$$y =_{df} (T_0; \langle 0, \bar{x}(i_0) \rangle; x(0), \lambda_0^*; x(1), \lambda_1^*; \dots; T_n; \langle 0, \bar{x}(i_n) \rangle; x(2n), \lambda_{2n}^*; x(2n+1), \lambda_{2n+1}^*; \dots)$$

is consistent with player I's w.s.  $s_+^\gamma$ . Since  $y$  is a win for I and  $\forall n \hat{t}_n = 0$ ,  $x \in C^*(\vec{A}) \subseteq (C, \vec{B})^*(\vec{A})$ .

Now suppose  $\forall n \neg C(x, n)$ . By the definition of  $s$ , for each  $n$ , a position

$$\hat{q}_{n+1} = (T_0; \langle 0, \bar{x}(i_0) \rangle; x(0), \lambda_0(n); x(1), \lambda_1(n); \dots; T_n; \langle 0, \bar{x}(i_n) \rangle; x(2n-2), \lambda_{2n}(n); x(2n+1), \lambda_{2n+1}(n))$$

exists which is consistent with  $s_+^\gamma$  and such that  $\lambda_1(n), \lambda_3(n), \lambda_5(n), \dots, \lambda_{2n+1}(n) \in C$ . Since  $\forall n \neg (C, \vec{B})(x, n)$ , show  $x \in D$  by integrating  $s_+^\gamma$  with respect to the ordinal auxiliary moves  $\lambda_{2i+1}$ 's (such integration is analogous to that described in Remark 2.13).  $x \in (C, \vec{B})^*(\vec{A}, D)$  since  $x \in D$  and  $\forall n \neg (C, \vec{B})(x, n)$ .

Thus, in any case,  $x$  is a win for I, and  $s$  is a w.s. for I. ■(Lemma 3.9)

Next we show the analogues of Lemmas 3.8 and 3.9 for the case where player II has w.s. for the auxiliary games of this section. We shall use the following simple observations:

(ix) If for some  $n$ ,  $\forall \ell \bar{x}(\ell) \notin R_n$  and  $x \notin A_n$ , then  $x \notin (C, \vec{B})^*(\vec{A}, D)$  and in particular  $x \notin (C, \vec{B})^*(\vec{A})$ .

(x) If  $\forall n \exists i_n \bar{x}(i_n) \in R_n$ , then  $\forall n \neg (\vec{B})(x, n)$ .

(ix) and (x) follow easily from (iii) and the definition of the  $R_j$ 's and  $A_n$ . (ix) is nothing more than a restatement of one direction of (iii).

**Lemma 3.10.** Player II has a w.s.  $s \in L(r) \left[ \vec{\#}_\gamma \right]$  for  $(C, \vec{B})^*(\vec{A})$  if he has one for  $G^\gamma$ .

**Proof:** Assume II has a w.s. for  $G^\gamma$ . Then  $s^\gamma \in L(r) \left[ \vec{\#}_\gamma \right]$  of Remark 3.4 is a w.s. for I. We define  $s$  from  $s^\gamma$  by simulating in the usual way the Borel auxiliary moves  $T_n$ 's.

**Construction of  $s$ :** Let

$$T_0 = \{ \text{positions } t \text{ in } G_A \mid \forall T \in L(r) \left[ \vec{\#}_\gamma \right] \langle 0, t \rangle \neq s^\gamma(T) \}.$$

Then

$$\forall t \in T_0 \forall T' \in L(r) \left[ \vec{\#}_\gamma \right] \langle 0, t \rangle \neq s^\gamma(T'). \quad (**)$$

Since  $s^\gamma \in L(r) \left[ \vec{\#}_\gamma \right]$ ,  $T_0 \in L(r) \left[ \vec{\#}_\gamma \right]$  so that by (\*\*)  $\langle 1, - \rangle = s^\gamma(T_0)$ .

$s_{p_0} \in L(\#_{(n)_0}(r, T_0))$ , where  $p_0 =_{df} (T_0; \langle 1, - \rangle)$ , is a w.s. for I in the game  $A_0(R_0; T_0; \emptyset)$ . Let  $s(p) = s_{p_0}(p)$  for any position  $p = (x(0); x(1); \dots; x(i-1))$  such that  $\forall j \leq i \bar{x}(j) \in T_0$ .

Suppose we reach a position such that  $\bar{x}(i_0) \notin T_0$  and  $\forall j < i_0 \bar{x}(j) \in T_0$ . We have defined

(xi)  $s(x(0); x(2); \dots; x(2j)) = x(2j+1)$  for  $2j < i_0$ .

Since  $\bar{x}(i_0) \notin T_0$ , there exists  $\tilde{T}_0 \in L(r) \left[ \vec{\#}_\gamma \right]$  such that  $\langle 0, \bar{x}(i_0) \rangle = s^\gamma(\tilde{T}_0)$ .

Let  $\langle \hat{t}_0, t_0 \rangle = \langle 0, \bar{x}(i_0) \rangle$  and  $p'_0 = (\tilde{T}_0; \langle 0, t_0 \rangle; x(0))$ . Define  $x(1) = s(x(0))$  to be  $s^\gamma(p'_0)$ . Let

$T_1 =_{df} \{ \text{positions } t \text{ in the game } \left( (C, \vec{B})^*(\vec{A}, D) \right) (\bar{x}(2), \bar{x}(i_0)) \mid \forall T' \in L(r) \left[ \vec{\#}_\gamma \right] \langle 0, t \rangle \neq s^\gamma(\tilde{T}_0; x(0); T') \}.$

Then  $T_1 \in L(r) \left[ \vec{\#}_\gamma \right]$  and  $\langle 1, - \rangle = s^\gamma \left( \tilde{T}_0; x(0); T_1 \right)$ . Let

$$p_1 =_{df} p'_0 \wedge (x(1); T_1; \langle 1, - \rangle)$$

and let  $s(p) = s_{p_1}(p)$  for any position  $p = (x(0); x(1); \dots, x(i-1))$  such that  $\forall j \leq i \bar{x}(j) \in T_1$ . This definition of  $s$  is consistent with (xi) since  $s_{p_1}$  is a w.s. for II in the game  $A_1(R_1, T_1; \bar{x}(i_{\max}(2)))$ . If  $\bar{x}(i) \in T_1$  holds at every position, then  $x \notin A_1(T_1; t_0, \bar{x}(i_{\max}(2)))$  so that  $x \notin A$ . If we reach a position such that  $\bar{x}(i_1) \notin T_1$ , then we continue to define  $s$  from  $s^\gamma$  in the same manner as above.

In general, suppose we reach a position such that

$$\bar{x}(i_j) \notin T_j \text{ and } \forall i < i_j \bar{x}(i) \in T_j \text{ for } j = 1, 2, 3, \dots, n-1.$$

Then let  $x(0), x(2), x(4), \dots, x(2n-2)$  and  $\tilde{T}_0, \tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_{n-1}$  be such that the position

$$p'_{n-1} = \left( \tilde{T}_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); \tilde{T}_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots \right. \\ \left. \dots; \tilde{T}_{n-1}; \langle 0, \bar{x}(i_{n-1}) \rangle; x(2n-2) \right)$$

is consistent with  $s^\gamma$ . Let

$$T_n =_{df} \{ \text{positions } t \text{ in the game } \left( (C, \vec{B})^* (\vec{A}, D) \right) (\bar{x}(i_{\max}(2n))) \mid$$

$$\forall T' \in L(r) \left[ \vec{\#}_\gamma \right] \langle 0, t \rangle \neq s^\gamma (p'_{n-1} \wedge (x(2n-1); T')) \}.$$

Then  $p_n = p'_{n-1} \wedge (x(2n-1); T_n; \langle 1, - \rangle)$  is consistent with  $s^\gamma$ .  $s_{p_n} \in L(\#_{(n)_0}(r, T_n))$  is a w.s. for II in the game  $A_n(R_n; T_n; \bar{x}(i_{\max}(2n)))$ . Let

$$s(p) = s_{p_n}(p)$$

for any position  $p = (x(0); x(1); \dots; x(i-1))$  such that  $\forall j \leq i \bar{x}(j) \in T_n$ . This completes our definition of  $s$ . Note that  $s \in L(r) \left[ \vec{\#}_\gamma \right]$ .

**Claim:** The strategy  $s$  of player II is a w.s. for the game  $(C, \vec{B})^* (\vec{A})$ .

Let  $x$  be a play of the game  $(C, \vec{B})^* (\vec{A})$  consistent with  $s$ . First consider the case in which there is a least  $n$  such that  $\forall i \bar{x}(i) \in T_n$ . Then for each  $j < n$ , there is a least  $i_j$  such that  $\bar{x}(i_j) \notin T_j$ . By the definition of  $x(2j+1) = s(x(0); x(2); \dots; x(2j))$  for  $j < n$  and by the definition of the  $T_i$ 's, there exist  $\tilde{T}_0, \tilde{T}_1, \tilde{T}_2, \dots, \tilde{T}_{n-1}$  such that the position

$$p_n = \left( \tilde{T}_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); \tilde{T}_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots; \tilde{T}_{n-1}; \langle 0, \bar{x}(i_{n-1}) \rangle; x(2n-2); x(2n-1); T_n; \langle 1, - \rangle \right)$$

is consistent with  $s^\gamma$ . Then  $\forall j < n - R_j(\bar{x}(i_j))$ , for otherwise  $p_n$  is a loss for II. Since  $s(p) = s_{p_n}(p)$  for any position  $p = (x(0); x(1); \dots; x(i-1))$  such that  $\forall j \leq i \bar{x}(j) \in T_n$  and we are assuming  $\forall j \bar{x}(j) \in T_n$ ,  $x$  is a play consistent with  $s_{p_n}$  so that

$$x \notin A_n(R_n; T_n; \bar{x}(i_{\max}(2n))).$$

Since  $\forall j \bar{x}(j) \in T_n$  and  $x$  is compatible with  $\bar{x}(i_{\max}(2n))$ ,  $x \notin A_n$  and  $\forall j \bar{x}(j) \notin R_n$ . Therefore, by (ix),  $x \notin (C, \vec{B})^*(\vec{A})$ .

Now assume for each  $n$ , there is a least  $i_n$  such that  $\bar{x}(i_n) \notin T_n$ . By the definition of  $x(2j+1) = s(x(0); x(2); \dots, x(2j))$  and by the definition of the  $T_i$ 's, there exist  $\tilde{T}_0, \tilde{T}_1, \tilde{T}_2, \dots$  such that the play

$$y =_{df} \left( \tilde{T}_0; \langle 0, \bar{x}(i_0) \rangle; x(0); x(1); \tilde{T}_1; \langle 0, \bar{x}(i_1) \rangle; x(2); x(3); \dots; \tilde{T}_n; \langle 0, \bar{x}(i_n) \rangle; x(2n); x(2n+1); \dots \right)$$

is consistent with  $s^\gamma$ . Since  $y$  is consistent with player II's w.s.  $s^\gamma$  and  $\forall n \hat{t}_n = 0$ ,  $\forall n R_n(\bar{x}(i_n))$  and  $x \notin C^*(\vec{A})$ . Therefore,  $x \notin (C, \vec{B})^*(\vec{A})$  and is a win for II since  $\forall n \neg(\vec{B})(x, n)$  and  $x \notin C^*(\vec{A})$ .

Consequently,  $s$  is a w.s. in  $(C, \vec{B})^*(\vec{A})$  of the player for whom  $s^\gamma$  is a w.s. ■(Lemma 3.10)

**Lemma 3.11.** Assume  $\#_{\gamma+1}(r)$  exists. Player II has a w.s.  $s \in L(\#_{\gamma+1}(r))$  for the game  $(C, \vec{B})^*(\vec{A}, D)$  if he has one for  $G_+^\gamma$ .

**Proof:** Let  $C$  be a class of indiscernibles for  $L(r) \left[ \overset{\vec{\gamma}}{\#_\gamma} \right]$ , closed and unbounded beneath every uncountable cardinal. Assume player II has a w.s. for  $G_+^\gamma$ . Then  $s_+^\gamma \in L(r) \left[ \overset{\vec{\gamma}}{\#_\gamma} \right]$  is a w.s. for II in  $G_+^\gamma$ . We construct  $s$  by:

- (i) simulating the Borel auxiliary moves with respect to  $s_+^\gamma$  analogous to the simulation (in Lemma 3.10) of such moves with respect to  $s^\gamma$ .
- (ii) integrating  $s_+^\gamma$  with respect to the ordinal auxiliary moves  $\lambda_i$ . (This integration is similar to that in Lemma 3.9.)

**Construction of  $s$ :** We proceed directly to the general case, since the construction is implicit from (i), (ii), and Lemmas 3.9 and 3.10.

Assume the position

$$p'_{n-1} =_{df} \left( \tilde{T}_0; \langle 0, \bar{x}(i_0) \rangle; x(0), \lambda_0(n-1); x(1), \lambda_1(n-1); \tilde{T}_1; \langle 0, \bar{x}(i_1) \rangle; x(2), \lambda_2(n-1); x(3), \lambda_3(n-1); \dots \right. \\ \left. \dots; \tilde{T}_{n-1}; \langle 0, \bar{x}(i_{n-1}) \rangle; x(2n-2), \lambda_{2n-2}(n-1) \right)$$

is consistent with  $s_+^\gamma$ , and  $\lambda_0(n-1), \lambda_2(n-1), \lambda_4(n-1), \dots, \lambda_{2n-2}(n-1) \in C$ . Define  $s(x(0); x(2); \dots; x(2n-2)) = x(2n-1)$  to be such that  $(x(2n-1), \lambda_{2n-1}(n-1)) = s_+^\gamma(p'_{n-1})$  for some  $\lambda_{2n-1}(n-1)$ . Let

Then  $T_n \in L(r) \left[ \overset{\vec{\gamma}}{\#_\gamma} \right]$  and  $\langle 1, - \rangle = s_+^\gamma(p_n)$ , where

$$p_n =_{df} p'_{n-1} \frown (x(2n-1), \lambda_{2n-1}(n-1); T_n; \langle 1, - \rangle).$$

$s_{p_n} \in L(\#_{(n)_0}(r, T_n))$  is a w.s. for II in the game  $A_n(R_n; T_n; \bar{x}(i_{\max}(2n)))$ . Set  $s(p) = s_{p_n}(p)$  for any position  $p = (x(0); x(1); x(2); \dots; x(i-1))$  compatible with  $\bar{x}(i_{\max}(2n))$  and such that  $\forall j \leq i \bar{x}(j) \in T_n$ .

Suppose we reach a position such that  $\bar{x}(i_n) \notin T_n$  and  $\forall j < i_n \bar{x}(j) \in T_n$ . We have defined

$$s(x(0); x(1); x(2); \dots; x(2j)) = x(2j+1) \text{ for } 2j+1 < i_{\max}(2n). \quad (\text{xii})$$

Since  $\bar{x}(i_n) \notin T_n$ , there exists  $\tilde{T}_n \in L(r) \left[ \vec{\#}_{\gamma} \right]$  such that

$$\langle 0, \bar{x}(i_n) \rangle = s_+^{\gamma} \left( p'_{n-1} \frown \left( x(2n-1), \lambda_{2n-1}; \tilde{T}_n \right) \right).$$

Fix  $x(2n)$  consistent with (xii). Select  $\lambda_0(n), \lambda_2(n), \lambda_4(n), \dots, \lambda_{2n}(n) \in C$  which are properly ordered with respect to  $\bar{x}(i_{\max}(2n+1))$ . Then  $\exists \lambda_1(n), \lambda_3(n), \lambda_5(n), \dots, \lambda_{2n+1}(n)$  such that the position

$$p'_n =_{df} \left( \tilde{T}_0; \langle 0, \bar{x}(i_0) \rangle; x(0), \lambda_0(n); x(1), \lambda_1(n); \tilde{T}_1; \langle 0, \bar{x}(i_1) \rangle; x(2), \lambda_2(n); x(3), \lambda_3(n); \dots \right. \\ \left. \dots; \tilde{T}_n; \langle 0, \bar{x}(i_n) \rangle; x(2n+1), \lambda_{2n+1}(n) \right)$$

is consistent with  $s_+^{\gamma}$ .

This completes our definition of  $s$ . Note that  $s \in L(\#_{\gamma+1}(r))$ .

**Claim.** The strategy  $s$  of player II is a w.s. for the game  $(C, \vec{B})^* (\vec{A}, D)$ .

Let  $x$  be a play of the game  $(C, \vec{B})^* (\vec{A}, D)$  consistent with  $s$ . First consider the case in which there is a least  $n$  such that  $\forall i \bar{x}(i) \in T_n$ . Then for each  $j < n$ , there is a least  $i_j$  such that  $\bar{x}(i_j) \notin T_j$ . By the definition of  $s$ , there is a w.s.  $s_{p_n}$  for II in the game  $A_n(R_n; T_n; \bar{x}(i_{\max}(2n)))$  such that:  $s(p) = s_{p_n}(p)$  for any position  $p = (x(0), x(1), \dots, x(i-1))$  compatible with  $\bar{x}(i_{\max}(2n))$  and such that  $\forall j \leq i \bar{x}(j) \in T_n$ . Since  $\forall i \bar{x}(i) \in T_n$ ,  $x$  is consistent with  $s_{p_n}$  so that  $x \notin A_n(R_n; T_n; \bar{x}(i_{\max}(2n)))$ . Hence, again since  $\forall i \bar{x}(i) \in T_n$ ,  $x \notin A_n$  and  $\forall i \bar{x}(i) \notin R_n$ . By (ix),  $x \notin (C, \vec{B})^* (\vec{A}, D)$ .

Now assume for each  $n$ , there is a least  $i_n$  such that  $\bar{x}(i_n) \notin T_n$ . By the definition of  $s$  and the  $T_n$ 's, there exists a sequence  $\langle \tilde{T}_n | n < \omega \rangle$  of elements from  $L(r) \left[ \vec{\#}_{\gamma} \right]$  such that:

For each  $n$ , there exist a sequence  $\vec{\lambda}(n) = \langle \lambda_{2j}(n) | j \leq n \rangle$  of elements from  $C$  and a position  $p'_n$  consistent with  $s_+^{\gamma}$ , where

$$p'_n = \left( \tilde{T}_0; \langle 0, \bar{x}(i_0) \rangle; x(0), \lambda_0(n); x(1), \lambda_1(n); \dots, \tilde{T}_n; \langle 0, \bar{x}(i_n) \rangle; x(2n), \lambda_{2n}(n); x(2n+1), \lambda_{2n+1}(n) \right).$$

Since each  $p'_n$  is consistent with  $s_+^{\gamma}$ ,  $R_n(\bar{x}(i_n))$  so that by (x),  $\forall n \neg (\vec{B})(x, n)$ .

First suppose  $\exists n C(x, n)$ . Then  $x$  is an element of each  $D_{\alpha}$  and by (vii), we select  $\langle \lambda_{2i}^* | i < \omega \rangle$  from  $C$  for which some play

$$y =_{df} \left( \tilde{T}_0; \langle 0, \bar{x}(i_0) \rangle; x(0), \lambda_0^*; x(1), \lambda_1^*; \dots, \tilde{T}_n; \langle 0, \bar{x}(i_n) \rangle; x(2n), \lambda_{2n}^*; x(2n+1), \lambda_{2n+1}^*; \dots \right).$$

is consistent with player II's w.s.  $s_+^{\gamma}$ .  $y$  is a win for II and  $\forall n \hat{t}_n = 0$  so that  $x \notin (C)^*(\vec{A})$ , and since  $\exists n C(x, n)$ ,  $x \notin (C, \vec{B})^*(\vec{A})$ .

Now suppose  $\forall n \neg C(x, n)$ . Since  $\forall n \neg (C, \vec{B})(x, n)$ , show  $x \notin D$  by integrating player II's w.s.  $s_+^\gamma$  with respect to the ordinal auxiliary moves  $\lambda_{2i}$ 's. Since  $\forall n \neg (C, \vec{B})(x, n)$  and  $x \notin D$ ,  $x \notin (C, \vec{B})^*(\vec{A}, D)$  and  $x$  is a win for II. ■(Lemma 3.11)

Theorem 3.1 and its corollaries now follow from Lemmas 3.8–3.11 and from the existence of  $s^\gamma$  and  $s_+^\gamma$ .