Now we prove that $\mathrm{Det}(\Pi^0_1, \Sigma^0_1)^*$ follows from $L(0^{\#\frac{1}{2}})[\#_1] \models$ "every real has a sharp." In the following proof, we integrate a w.s. for an auxiliary game G^2. G^2 contains the moves of the auxiliary games G^0 and G^1. Furthermore, we shall use Theorems 1.0 and 1.1 to obtain a w.s. for G^2_p whenever p is a legal position of G^2 such that the moves of G^2 following p constitute a play of either $G^0(\vec{U};\vec{u})$ or $G^1(\vec{U};\vec{u})$ for some $\vec{U}$ and $\vec{u}$.

Theorem 1.2. If $L(0^{\#\frac{1}{2}})[\#_1] \models$ "every real has a sharp," then

$$\mathrm{Det}(\Pi^0_1, \Sigma^0_1)^*.$$

Proof: Assume $L(0^{\#\frac{1}{2}})[\#_1] \models$"every real has a sharp." Let

$$B \in \Pi^0_1,\ C \in \Sigma^0_1, \text{ and } \langle A_\alpha \mid \alpha < \omega^2 \rangle$$

strongly witness $A \in (\Pi^0_1, \Sigma^0_1)^*$. There exist R_B and R_C in Δ^0_1 such that

i.) $B(x,n) \leftrightarrow \forall k R_B(\bar{x}(k), n)$;

ii.) $C(x,n) \leftrightarrow \exists k R_C(\bar{x}(k), n)$;

iii.) if $\neg R_B(\bar{x}(k), n)$ and $\forall j < k R_B(\bar{x}(j), n)$, then k is odd; and

iv.) if $R_C(\bar{x}(k), n)$, and $\forall j < k \neg R_C(\bar{x}(j), n)$, then k is odd.

We show that G_A has a w.s. s. Conditions (iii) and (iv) help to simplify the proof.

We describe an open game G^2 which has a w.s. $s^2 \in L(0^{\#\frac{1}{2}})[\#_1]$. We integrate s^2 to get the w.s. $s \in L(0^{\#\frac{1}{2}})[\#_1]$ for G_A. The moves of G^2 are the integer moves $x(i)$ of G_A as well as two types of auxiliary moves: Borel auxiliary moves and ordinal auxiliary moves. In G^2, first the players play Borel auxiliary moves Q and $\langle \hat{q}, q\rangle$ which are determined by the Σ^0_1 set $\{q|\exists k R_C(q,k)\}$:

$$G^2 \quad \begin{array}{lll} \mathrm{I} & Q & \\ \mathrm{II} & & \langle\hat{q},q\rangle \end{array} \ \cdots.$$

Let $\vec{Q} = \langle Q|\hat{q} = 1\rangle$ and $\vec{q} = \langle q|\hat{q} = 0\rangle$. G^2 contains a sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle)|\forall j < n\ \hat{t}_j = 0\rangle$ of Borel auxiliary moves. The sequence

$$\langle (T_n; \langle \hat{t}_n, t_n\rangle)|\forall j < n\ \hat{t}_j = 0\rangle$$

and the Π^0_1 set B are related via R_B, and player I may only play $T_i \in L(\vec{Q})[\#_1]$. If II plays $\hat{q} = 1$, then the moves of G^2 following $\langle 1, -\rangle$ constitute a play of $G^0(Q)$. If II plays $\hat{q} = 0$, then the moves of G^2 following $\langle 0, q\rangle$ constitute a play of $G^1(\vec{Q}; \vec{q})$ (i.e. of $G^1(q)$).

If $\hat{q} = 1$ and all $\hat{t}_i = 0$, then the play of G^2 is

$$G^2 \quad \begin{array}{lllllllllll} \mathrm{I} & Q & & T_0 & & x(0) & & T_1 & & x(2) & T_2 & & x(4) & \\ \mathrm{II} & & \langle\hat{q},q\rangle & & \langle 0,t_0\rangle & & x(1) & & \langle 0,t_1\rangle & x(3) & & \langle 0,t_2\rangle & & x(5) \end{array} \ \cdots.$$

If $\hat{q} = 0$ and all $\hat{t}_i = 0$, then the play of G^2 is

$$G^2 \quad \begin{array}{llllllllllllll} \mathrm{I} & Q & & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & T_2 & & x(4),\lambda_4 & \\ \mathrm{II} & & \langle\hat{q},q\rangle & & \langle 0,t_0\rangle & & x(1),\lambda_1 & & \langle 0,t_1\rangle & x(3),\lambda_3 & & \langle 0,t_2\rangle & & x(5),\lambda_5 \end{array} \ \cdots.$$

Whenever II plays $\hat{q} = 1$ and some $\hat{t}_n = 1$, a typical play of G^2 is

$$G^2 \quad \begin{array}{llllllllll} \mathrm{I} & Q & & T_0 & & x(0) & & T_1 & & x(2) & \\ \mathrm{II} & & \langle\hat{q},q\rangle & & \langle 0,t_0\rangle & & x(1) & & \langle 0,t_1\rangle & & x(3) \end{array} \ \cdots$$

$$\cdots \begin{array}{llllllllll} T_{n-1} & & x(2n-2), & & T_n & & x(2n),\xi_0 & & x(2n+2),\xi_2 & \\ & \langle 0,t_{n-1}\rangle & & x(2n-1) & & \langle 1,-\rangle & & x(2n+1),\xi_1 & & x(2n+3),\xi_3 \end{array} \ \cdots.$$

Whenever II plays $\hat{q} = 0$ and some $\hat{t}_n = 1$, a typical play of G^2 is

$$G^2 \quad \begin{array}{llllllllll} \mathrm{I} & Q & & T_0 & & x(0),\lambda_0 & & T_1 & & x(2),\lambda_2 & \\ \mathrm{II} & & \langle\hat{q},q\rangle & & \langle 0,t_0\rangle & & x(1),\lambda_1 & & \langle 0,t_1\rangle & & x(3),\lambda_3 \end{array} \ \cdots$$

$$\cdots \begin{array}{llllllllll} T_{n-1} & & x(2n-2),\lambda_{2n-2} & & T_n & & x(2n),\xi_0 & & x(2n+2),\xi_2 & \\ & \langle 0,t_{n-1}\rangle & & x(2n-1),\lambda_{2n-1} & & \langle 1,-\rangle & & x(2n+1),\xi_1 & & x(2n+3),\xi_3 \end{array} \ \cdots.$$

Let μ be least such that $R_C(q,\mu)$ whenever $\hat{q} = 0$; otherwise μ is undefined. If II plays $\langle \hat{q}, q\rangle = \langle 0, q\rangle$, ordinal auxiliary moves λ_i's are played and the λ_i's are properly ordered with respect to $\bar{x}(i+1)$ and $\langle A_\alpha | \alpha < \omega \cdot (\mu+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(0))} | i \leq \mu\rangle$.

Regardless of what II plays for $\hat{q}$, if II plays $\hat{t}_n = 1$, then ordinal auxiliary moves ξ_i's are played so that the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n+1)\rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T_n))} | i \leq n\rangle$.

Player I wins G^2 iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. G^2 is an open game. Therefore, define for each ordinal α, P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in ON} P_\alpha$. If p is a legal position in G^2, let ℓ_p denote the set of legal positions in G^2 consistent with p. The set of legal positions for G^2 is in $L(0^{\#_2^1})[\#_1]$. Moreover, whenever p is a legal position in G^2, the following holds:

v.) If p includes the move $\langle \hat{q}, q\rangle$, then $L(\vec{Q})[\#_1]$ contains ℓ_p. Furthermore, Q is coded by a real in $L(\#_2^1(0))[\#_1]$.

vi.) If p includes moves $\langle \hat{q}, q\rangle$ and $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $\ell_p \in L(\vec{Q}, T_n)$ and T_n is coded by a real in $L(\vec{Q})[\#_1]$.

Using $\langle P_\alpha | \alpha \in \mathrm{ON}\rangle$, define a wellordering $\prec$ of the legal positions for G^2 such that $\prec$ is definable in $L(0^{\#_2^1})[\#_1]$ and whenever p is a legal position in G^2, $\prec |\ell_p$ is a wellordering of the legal positions of G^2 consistent with p and the following hold:

vii.) If p includes the move $\langle \hat{q}, q\rangle = \langle 1, -\rangle$, then $\vec{Q} = \langle Q\rangle$ and $\prec |\ell_p$ is definable in $L(\vec{Q})[\#_1]$. If p includes the move $\langle 0, q\rangle$, then $\vec{Q} = \langle\ \rangle$ and $\prec |\ell_p$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu \rangle$. (Recall that if $\hat{q} = 0$, μ is least such that $R_C(q, \mu)$.)

viii.) If p includes moves $\langle \hat{q}, q \rangle$ and $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $\prec |\ell_p$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{Q}, T_n))} | i \leq n \rangle$.

By Lemma 0.14, use $\prec$ to define the canonical w.s. s^2 for G^2. Then s^2 is definable in $L(0^{\#_2^1})[\#_1]$. Furthermore, if p is a legal position in G^2, then $s^2 | \ell_p$ is a w.s. for G_p^2 and is definable in any inner model of ZF in which $\prec |\ell_p$ is definable. Therefore, s^2 has the following properties:

Lemma 1.2.1. Let p be a legal position in G^2. Recall that whenever $\hat{q} = 0$, μ is least such that $R_C(q, \mu)$ and otherwise μ is undefined. Then $s^2 | \ell_p$ is a w.s. for G_p^2 and each of the following hold:

ix.) If p includes the move $\langle \hat{q}, q \rangle = \langle 1, - \rangle$, then $s^2 | \ell_p$ is definable in $L(\vec{Q})[\#_1]$. If p includes the move $\langle \hat{q}, q \rangle = \langle 0, q \rangle$, then $s^2 | \ell_p$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu \rangle$.

x.) If p includes moves $\langle \hat{q}, q \rangle$ and $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $s^2 | \ell_p$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}))} | i \leq n \rangle$.

If p is a legal position in G^2 which includes the move $\langle \hat{q}, q \rangle = \langle 0, q \rangle$, we use Property (ix) and indiscernibles for $L(\vec{Q})[\#_1]$ to integrate $s^2 | \ell_p$ with respect to the λ_i's. If p is a legal position in G^2 which includes moves $\langle \hat{q}, q \rangle$ and $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $\vec{Q}$ and T_n are coded by a real in $L(\vec{Q})[\#_1]$ and we use Property (x) and indiscernibles for $L(\vec{Q}, T_n)$ to integrate $s^2 | \ell_p$ with resepct to the ξ_i's.

Claim I: Player I has a w.s. for G_A if he has one for G^2.

Let's first consider the case in which $\langle\ \rangle \in P$. Then $s^2 \in L(0^{\#^1_2})[\#_1]$ is a w.s. for I in G^2. We use s^2 to define a w.s. s for I in G_A. Define Q so that the position $p_0 = (Q; \langle 1, -\rangle)$ is consistent with s^2. By Lemma 1.2.1(ix), $s^2|\ell_{p_0}$ is definable in $L(\vec{Q})[\#_1]$. By Theoerm 1.0, integrate $s^2|\ell_{p_0}$ so as to obtain a w.s. $s_0 \in L(\vec{Q})[\#_1]$ for $A(\vec{Q})$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that $\forall i' \leq i \forall k \neg R_C(\bar{x}(i'), k)$.

Suppose we reach a position such that $\exists i \exists k R_C(\bar{x}(i), k)$. Let i be least such that $\exists k R_C(\bar{x}(i), k)$, and let μ be least such that $R_C(\bar{x}(i), \mu)$. By (iv), i is odd. Let $Q = s^2(\ )$ (as above), $\langle \hat{q}, q\rangle = \langle 0, \bar{x}(i)\rangle$, and $p_1 = (Q; \langle 0, q\rangle)$. By Lemma 1.2.1(ix), $s^2|\ell_{p_1}$ is definable in $L[\#_1]$ from $\langle \omega_{i+1}^{L(\#^1_2(0))} | i \leq \mu\rangle$. Since $0^{\#^1_2}$ exists, by Theorem 1.1, integrate $s^2|\ell_{p_1}$ so as to obtain a w.s. $s_1 \in L(0^{\#^1_2})$ for $A(q)$. Let $s(p) = s_1(p)$ for any position p which extends q.

Claim: The strategy s of player I is a w.s. in G_A.

Suppose x is a play of G_A consistent with s. By the definition of s, there exists a w.s. $s_0 \in L(\vec{Q})[\#_1]$ for $A(Q)$ such that $x(2i) = s_0(x(1); x(3); ...; x(2i-1))$ whenever $\forall i' \leq 2i \forall n \neg R_C(\bar{x}(i'), n)$. If $\forall i \forall n \neg R_C(\bar{x}(i), n)$ holds, then $x \in A(Q)$ so that $x \in A$ by Theorem 1.0.

Otherwise, $\exists n R_C(\bar{x}(i), n)$ for some least i. By the definition of s, there exists a w.s. $s_1 \in L(0^{\#^1_2})$ for $A(\bar{x}(i))$ such that

$$x(2k) = s_1(x(1); x(3); ...; x(2k-1)) \text{ whenever } 2k > i.$$

Therefore, $x \in A(\bar{x}(i))$ so that $x \in A$. Thus, s is a win for I.

Claim II: Player II has a w.s. for G_A if he has one for G^2.

Now let's consider the case $\langle\ \rangle \notin P$. We integrate II's w.s. s^2 for G^2 to get the w.s. $s \in L(0^{\#_{\frac{1}{2}}})[\#_1]$ for II in G_A. Let

$$Q = \{\text{positions } q \text{ in } G_A \mid \forall Q' \in L(\#_{\frac{1}{2}}(0))[\#_1]\ \langle 0, q\rangle \neq s^2(Q')\}.$$

Then $Q \in L(0^{\#_{\frac{1}{2}}})[\#_1]$ and $\langle 1, -\rangle = s^2(Q)$. Let $p_0 = (Q; \langle 1, -\rangle)$.

By Lemma 1.2.1(ix), $s^2|\ell_{p_0}$ is a w.s. for $A(Q)$ and is definable in $L(Q)[\#_1]$. By Theorem 1.0, there exists a w.s. $s_0 \in L(Q)[\#_1]$ for the game $A(Q)$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that $\forall i' \leq i\ \bar{x}(i') \in Q$.

Suppose we reach a position $\bar{x}(i)$ (of least length) such that $\bar{x}(i) \notin Q$. Then i is odd and there exists $Q' \in L(0^{\#_{\frac{1}{2}}})[\#_1]$ such that the position $p_1 = (Q'; \langle 0, \bar{x}(i)\rangle)$ is consistent with s^2. By Theorem 1.1, obtain a w.s. $s_1 \in L(\#_{\frac{1}{2}}(0))$ for $A(\bar{x}(i))$. Let $s(p) = s_1(p)$ for any position p which extends $\bar{x}(i)$.

Claim: The strategy s of player II is a w.s. for G_A.

Suppose x is a play of G_A consistent with s. By the definition of s, there exists a w.s. $s_0 \in L(Q)[\#_1]$ for $A(Q)$ such that $x(2i+1) = s_0(x(0); x(2); ...; x(2i))$ whenever $\forall i' \leq 2i+1\ \bar{x}(i') \in Q$. If $\forall i\ \bar{x}(i) \in Q$, then $x \notin A(Q)$ so that $x \notin A$.

Otherwise, there exists i such that $\bar{x}(i) \notin Q$ and $\forall j < i\ \bar{x}(j) \in Q$. By the definition of s, there exists a w.s. $s_1 \in L(\#_{\frac{1}{2}}(0))$ for $A(\bar{x}(i))$ such that

$x(2k+1) = s_1(x(0); x(2); ...; x(2k))$ whenever $2k+1 \geq i$. Therefore, since s_1 is a w.s. for II, $x \notin A(\bar{x}(i))$ so that $x \notin A$. Consequently, s is a w.s. in G_A of the player for whom s^2 is a w.s. ∎

Now we show that the existence of indiscernibles for $L(0^{\#_2^1})[\#_1^1]$ implies the determinacy of $(\Pi_1^0, \Sigma_1^0)^*_+$. The proof of this theorem is almost identical to the proof of Theorem 1.2.

Theorem 1.3. If $0^{2\#_2^1}$ exists (i.e. $L(0^{\#_2^1})[\#_1]$ has indiscernibles), then $\mathrm{Det}(\Pi_1^0, \Sigma_1^0)^*_+$.

Proof: Assume $L(0^{\#_2^1})[\#_1]$ has an uncountable set C_2^1 of indiscernibles. Let $B \in \Pi_1^0$, $C \in \Sigma_1^0$, $\langle \mathrm{A}_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ strongly witness $\mathrm{A} \in (\Pi_1^0, \Sigma_1^0)^*_+$. Then there exist R_B and R_C in Δ_1^0 such that

i.) $B(x, n) \leftrightarrow \forall k R_B(\bar{x}(k), n)$;

ii.) $C(x, n) \leftrightarrow \exists k R_C(\bar{x}(k), n)$;

iii.) if $\neg R_B(\bar{x}(k), n)$ and $\forall j < k R_B(\bar{x}(j), n)$, then k is odd; and

iv.) if $R_C(\bar{x}(k), n)$ and $\forall j < k \neg R_C(\bar{x}(j), n)$, then k is odd.

Conditions (iii) and (iv) help to simplify the proof.

We describe an open game G^3 which has a w.s. $s^3 \in L(0^{\#_2^1})[\#_1]$. We complete the proof by integrating s^3 to get the w.s. $s \in L(0^{2\#_2^1})$ for G_A. The moves of G^3 are the same as the moves of G^2 with one exception: In G^3, ordinal auxiliary moves λ_{2i} and λ_{2i+1} are respectively played with integer moves $x(2i)$ and $x(2i+1)$ whenever $\hat{q} = 1$; whereas, no ordinal auxiliary moves

λ_i are played in G^2 whenever II plays $\hat{q} = 1$.

If all $\hat{t}_i = 0$, then the play of G^3 is

$$\begin{array}{lccccccccc} \text{I} & Q & T_0 & x(0),\lambda_0 & T_1 & x(2),\lambda_2 & T_2 & x(4),\lambda_4 & \\ \text{II} & \langle\hat{q},q\rangle & \langle 0,t_0\rangle & x(1),\lambda_1 & \langle 0,t_1\rangle & x(3),\lambda_3 & \langle 0,t_2\rangle & x(5),\lambda_5 & \ldots . \end{array}$$

If some $\hat{t}_n = 1$, the play of G^3 is

$$\begin{array}{lcccccc} \text{I} & Q & T_0 & x(0),\lambda_0 & T_1 & x(2),\lambda_2 & \\ \text{II} & \langle\hat{q},q\rangle & \langle 0,t_0\rangle & x(1),\lambda_1 & \langle 0,t_1\rangle & x(3),\lambda_3 & \cdots \end{array}$$

$$\begin{array}{cccccccc} \cdots & T_{n-1} & x(2n-2),\lambda_{2n-2} & T_n & x(2n),\xi_0 & x(2n+2),\xi_2 & \\ & \langle 0,t_{n-1}\rangle & x(2n-1),\lambda_{2n-1} & \langle 1,-\rangle & x(2n+1),\xi_1 & x(2n+3),\xi_3 & \cdots . \end{array}$$

The moves of G^3 following $\langle \hat{q}, q\rangle$ constitute a play of $G^1(q)$ whenever II plays $\hat{q} = 0$, and they constitute a play of $G^1(Q)$ whenever II plays $\hat{q} = 1$. Let $\vec{Q} = \langle Q \mid \hat{q} = 1\rangle$. Player I must play so that $Q \in L(0^{\#^1_2})[\#_1]$ and each $T_i \in L(\vec{Q})[\#_1]$. Moreover, the Borel auxiliary moves must satisfy the same conditions as in G^2: $\{q|\exists k R_C(q, k)\}$ determines the Borel auxiliary moves Q and $\langle \hat{q}, q\rangle$, and the sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle)|\forall j < n\ \hat{t}_j = 0\rangle$ of Borel auxiliary moves and the Π^0_1 set B are related via R_B.

If $\hat{q} = 0$, let μ be least such that $R_C(q, \mu)$; otherwise, let $\mu = m$. The λ_i's are properly ordered with respect to $\langle A_\alpha|\alpha < \omega\cdot(\mu+1)\rangle$ using $\langle \omega_{i+1}^{L(\#^1_2(\vec{Q}))}|i \leq \mu\rangle$. (Notice $\#^1_2(\vec{Q})$ can equal $2\#^1_2(0)$.) If $\hat{t}_n = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha|\alpha < \omega \cdot (n + 1)\rangle$ using $\langle \omega_{i+1}^{L(\#^1_1(\vec{Q},T_n))}|i \leq n\rangle$.

Player I wins G^3 iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. G^3 is an open game and therefore we define, for each ordinal α, P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha\in ON} P_\alpha$. If p is a legal position in G^3, let ℓ_p denote the set of legal positions in G^3 consistent with p. The set of legal positions for G^3 is in $L(0^{\#_2^1})[\#_1]$. Moreover, whenever p is a legal position of G^3, the following properties hold:

v.) If p includes the move $\langle \hat{q}, q\rangle$, then $L(\vec{Q})[\#_1]$ contains ℓ_p. Furthermore, Q is coded by a real in $L(2\#_2^1(0))$.

vi.) If p includes moves $\langle \hat{q}, q\rangle$ and $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $\ell_p \in L(\vec{Q}, T_n)$ and $\vec{Q}$ and T_n are coded by a real in $L(\vec{Q})[\#_1]$.

$L(0^{\#_2^1})[\#_1]$ has a definable wellordering $\prec$ of the legal positions for G^3 such that whenever p is a legal position in G^3, $\prec |\ell_p$ is a wellordering of the legal positions of G^3 consistent with p and the following hold:

vii.) If p includes moves $\langle 0, q\rangle$, then $\prec |\ell_p$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu\rangle$. (Recall that if $\hat{q} = 0$, μ is least such that $R_C(q, \mu)$, and otherwise $\mu = m$.)

viii.) If p includes moves $\langle \hat{q}, q\rangle$ and $\langle \hat{t}_n, t_n\rangle = \langle 1, -\rangle$, then $\prec |\ell_p$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T_n))} | i \leq n\rangle$.

By Lemma 0.14, use $\prec$ to define s^3 to be the canonical w.s. for G^3. Then s^3 is definable in $L(0^{\#_2^1})[\#_1]$. Furthermore, if p is a legal position in G^3, then $s^3|\ell_p$ is a w.s. for G^3_p in any inner model of ZF in which $\prec |\ell_p$ is definable. Therefore, s^3 has the following properties:

Lemma 1.3.1. Let p be a legal position in G^3. Recall that whenever $\hat{q} = 0$, μ is least such that $R_C(q, \mu)$ and otherwise $\mu = m$. Then $s^3|\ell_p$ is a w.s. for G^3_p and each of the following hold:

ix.) If p includes the move $\langle \hat{q}, q \rangle$, then $s^3|\ell_p$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu \rangle$.

x.) If p includes the move $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $s^3|\ell_p$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T_n))} | i \leq n \rangle$.

If p is a legal position in G^3 which includes the move $\langle \hat{q}, q \rangle$, we use Property (ix) and indiscernibles for $L(\vec{Q})[\#_1]$ to integrate $s^3|\ell_p$ with respect to the λ_i's. If p is a legal position in G^3 which includes the moves $\langle \hat{q}, q \rangle$ and $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then T_n is coded by a real in $L(\vec{Q})[\#_1]$ and we use Property (x) and indiscernibles for $L(\vec{Q}, T_n)$ to integrate $s^3|\ell_p$ with resepct to the ξ_i's.

Claim I: Player I has a w.s. for G_A if he has one for G^3.

Let's first consider the case in which $\langle \, \rangle \in P$. Then $s^3 \in L(0^{\#_2^1})[\#_1]$ is a w.s. for I in G^3. We use s^3 to define a w.s. s for I in G_A. Let $Q = s^3(\langle \, \rangle)$, $\langle \hat{q}, q \rangle = \langle 1, - \rangle$, and $p_0 = (Q; \langle 1, - \rangle)$. By Lemma 1.3.1(ix), $s^3|\ell_{p_0}$ is definable in $L(Q)[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(Q))} | i \leq m \rangle$. Since $L(Q)[\#_1]$ has indiscernibles, by Theorem 1.1 obtain a w.s. $s_0 \in L(\#_2^1(Q))$ for $A(Q)$ by integrating $s^3|\ell_{p_0}$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that $\forall i' \leq i \forall n \neg R_C(\bar{x}(i'), n)$.

Suppose we reach a position such that $\exists i \exists n R_C(\bar{x}(i), n)$. Let i be least such that $\exists n R_C(\bar{x}(i), n)$. Let $Q = s^3(\ )$ (as above), $\langle \hat{q}, q \rangle = \langle 0, \bar{x}(i) \rangle$, and $p_1 = (Q; \langle 0, q \rangle)$. By Lemma 1.3.1(ix), $s^3|\ell_{p_0}$ is definable in $L[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(0))} | i \leq \mu \rangle$. Since $L[\#_1]$ has indiscernibles, by Theorem 1.1, integrate

$s^3|\ell_{p_1}$ so as to obtain a w.s. $s_1 \in L(0^{\#^1_2})$ for $A(q)$. Let $s(p) = s_1(p)$ for any position p which extends q.

Claim: The strategy s of player I is a w.s. in G_A.

Suppose x is a play of G_A consistent with s. By the definition of s, there exists a w.s. $s_0 \in L(\#^1_2(Q))$ for $A(Q)$ (where $Q = s^3(\ )$) such that $x(2i) = s_0(x(1); x(3); ...; x(2i-1))$ whenever $\forall i' \leq 2i \forall n \neg R_C(\bar{x}(i'), n)$. If $\forall i \forall n \neg R_C(\bar{x}(i), n)$ holds, then $x \in A(Q)$ so that $x \in A$ by Theorem 1.1.

Otherwise, $\exists n R_C(\bar{x}(i), n)$ for some least i. By the definition of s, there exists a w.s. $s_1 \in L(0^{\#^1_2})$ for $A(\bar{x}(i))$ such that

$$x(2k) = s_1(x(1); x(3); ...; x(2k-1)) \text{ whenever } 2k > i.$$

Therefore, $x \in A(\bar{x}(i))$ so that $x \in A$. Thus, s is a win for I.

Claim II: Player II has a w.s. for G_A if he has one for G^3.

Now let's consider the case $\langle\ \rangle \notin P$. We integrate II's w.s. s^3 for G^3 to get the w.s. $s \in L(0^{2\#^1_2})$ for II in G_A. Let

$$Q = \{\text{positions } q \text{ in } G_A | \forall Q' \in L(\#^1_2(0))[\#_1]\ \langle 0, q\rangle \neq s^3(Q')\}.$$

Then $\langle 1, -\rangle = s^3(Q)$ and $Q \in L(0^{\#^1_2})[\#_1]$. Let $p_0 = (Q; \langle 1, -\rangle)$. By Lemma 1.3.1(ix), $s^3|\ell_{p_0}$ is a w.s. for $A(Q)$ and is definable in $L(Q)[\#_1]$ from $\langle \omega_{i+1}^{L(\#^1_2(Q))} | i \leq m\rangle$. Since indiscernibles for $L(Q)[\#_1]$ exist, by Theorem 1.1, integrate $s^3|\ell_{p_0}$ so as to obtain a w.s. $s_0 \in L(\#^1_2(Q))$ for the game $A(Q)$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); ...; x(i-1))$ such that $\forall i' \leq i\ \bar{x}(i') \in Q$.

Suppose we reach a position $\bar{x}(i)$ (of least length) such that $\bar{x}(i) \notin Q$. Then i is odd and there exists $Q' \in L(0^{\#_2^1})[\#_1]$ such that the position $p_1 = (Q'; \langle 0, \bar{x}(i) \rangle)$ is consistent with s^3. By Theorem 1.1, obtain a w.s. $s_1 \in L(\#_1^1(0))$ for $A(\bar{x}(i))$ by appropriately integrating $s^3|\ell_{p_1}$. Let $s(p) = s_1(p)$ for any position p which extends $\bar{x}(i)$.

Claim: The strategy s of player II is a w.s. for G_A.

Suppose x is a play of G_A consistent with s. By the definition of s, there exists a w.s. $s_0 \in L(\#_2^1(Q))$ for $A(Q)$ such that

$$x(2i) = s_0(x(1); x(3); ...; x(2i-1)) \text{ whenever } \forall i' \leq 2i \; \bar{x}(i') \in Q.$$

If $\forall i \; \bar{x}(i) \in Q$, then $x \notin A(Q)$ so that $x \notin A$.

Otherwise, there exists i such that $\bar{x}(i) \notin Q$ and $\forall j < i \; \bar{x}(j) \in Q$. By the definition of s, there exists a w.s. $s_1 \in L(\#_2^1(0))$ for $A(\bar{x}(i))$ such that $x(2k+1) = s_1(x(0); x(2); ...; x(2k))$ whenever $2k+1 \geq i$. Therefore, since s_1 is a w.s. for II, $x \notin A(\bar{x}(i))$ so that $x \notin A$. Consequently, s is a w.s. in G_A of the player for whom s^3 is a w.s. ∎

Definition 1.2. Let $B \in \Pi_1^0$, $C \in \Sigma_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ strongly witness $A \in (\Pi_1^0, \Sigma_1^0)_+^*$. Then we refer to the auxiliary game G^3 described in the Proof of Theorem 1.3 as *the G^3 auxiliary game determined by $B \in \Pi_1^0$, $C \in \Sigma_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$.*

Suppose $\vec{U} = \langle U_i | i < \eta \rangle$ and $\vec{u} = \langle u_i | i < \theta \rangle$ respectively are a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of

G_A. Then *the* $G^3(\vec{U};\vec{u})$ *auxiliary game determined by* B, C, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, *and* $m \in \omega$ is the game which has exactly the same moves as G^3, player I wins iff a position is reached at which player II cannot make a (legal) move, and the moves are subject to the following conditions:

i.) Each $\bar{x}(i) \in \bigcap_{i<\eta} U_i$ and each $\bar{x}(i)$ must be consistent with every u_i.

ii.) $Q \in L(\#^1_2(\vec{U}))[\#_1]$ (where $\vec{Q} = \langle Q \mid \hat{q} = 1\rangle$).

iii.) $T_i \in L(\vec{U},\vec{Q})[\#_1]$.

iv.) The λ_i's are properly ordered with repect to $\langle A_\alpha | \alpha < \omega \cdot (\mu+1)\rangle$ using $\langle \omega_{i+1}^{L(\#^1_2(\vec{U},\vec{Q}))} | i \leq \mu\rangle$ (where μ is least such that $R_C(q,\mu)$ if $\hat{q} = 0$ and otherwise $\mu = m$). (Notice $\#^1_2(\vec{U},\vec{Q})$ can equal $2\#^1_2(\vec{U})$.)

v.) If $\hat{t}_n = 1$, the ξ_i's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega\cdot(n+1)\rangle$ using $\langle \omega_{i+1}^{L(\#^1_1(\vec{U},\vec{Q},T_n))} | i \leq n\rangle$.

vi.) $\{q | \exists k R_C(q,k)\}$ determines the Borel auxiliary moves Q and $\langle \hat{q}, q\rangle$.

vii.) The sequence $\langle (T_n; \langle \hat{t}_n, t_n\rangle) | \forall j < n\ \hat{t}_j = 0\rangle$ of Borel auxiliary moves and the Π^0_1 set B are related via R_B.

These conditions are analogous to the conditions for the moves of G^3. The last two are conditions which the moves of G^3 must satisfy. The other conditions are derived by changing the conditions for the moves of G^3 so that they are consistent with $\vec{U}$ and $\vec{u}$. We refer to $G^3(\vec{U};\vec{u})$ instead of the $G^3(\vec{U};\vec{u})$ auxiliary game determined by B, C, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ whenever B, C, $\langle A_\alpha \mid \alpha < \omega^2\rangle$, and $m \in \omega$ are clear from the context. Analogous to

Theorem 1.3, we have the following:

Corollary 1.3.1. Let B, C, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, m, A, $\vec{U}$, and $\vec{u}$ be as in Definition 1.2. Let p be a legal position of a game G^* such that the moves of G^* following p constitute a play of $G^3(\vec{U}; \vec{u})$. Suppose $2\#_2^1(\vec{U})$ exists, $\vec{U}$ has a wellordering definable in $L(\vec{U})$, and s^* is a w.s. for G^* such that $s^*|\ell_p \in L(\#_2^1(\vec{U}))[\#_1]$. Then $s^*|\ell_p$ can be integrated so as to obtain a w.s. $s_p \in L(2\#_2^1(\vec{U}))$ for $A(\vec{U}; \vec{u})$ such that the following hold:

i.) s_p is a w.s. of the player for whom s^* is a w.s.

ii.) If s^* is a w.s. for I, $\hat{p}$ is a position consistent with s_p, and the moves in $\hat{p}$ of player II are consistent with $\vec{u}$, then $\hat{p} \in \bigcap_{i<\eta} U_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with s_p, then $x \in A(\vec{u})$.

iii.) Let $\hat{p}$ be a position consistent with s_p and with $\vec{U}$. If the moves in $\hat{p}$ of the player for whom s_p is not a w.s. are consistent with $\vec{u}$, then $\hat{p}$ is consistent with $\vec{u}$. ∎