

Now we show that $\text{Det}(\Pi_1^0, 2 * \Sigma_1^0)^*$ follows from $L(0^{2\#_2^1})[\#_1] \models$ “every real has a sharp.” The proof of this theorem is similar to the proof of Theorem 1.2.

Theorem 1.4. If $L(0^{2\#_2^1})[\#_1] \models$ “every real has a sharp,” then

$$\text{Det}(\Pi_1^0, 2 * \Sigma_1^0)^*.$$

Proof: Assume $L(0^{2\#_2^1})[\#_1] \models$ “every real has a sharp.” Let

$$B \in \Pi_1^0, C_1, C_2 \in \Sigma_1^0, \text{ and } \langle A_\alpha \mid \alpha < \omega^2 \rangle$$

strongly witness $A \in (\Pi_1^0, 2 * \Sigma_1^0)^*$. Wlog $C_1 \subseteq C_2$. There exist R_B, R_{C_1} , and R_{C_2} in Δ_1^0 such that

- i.) $B(x, n) \leftrightarrow \forall k R_B(\bar{x}(k), n)$;
- ii.) $C_i(x, n) \leftrightarrow \exists k R_{C_i}(\bar{x}(k), n)$;
- iii.) if $\neg R_B(\bar{x}(k), n)$ and $\forall j < k R_B(\bar{x}(j), n)$, then k is odd; and
- iv.) if $i \in \{1, 2\}$, $R_{C_i}(\bar{x}(k), n)$, and $\forall j < k \neg R_{C_i}(\bar{x}(j), n)$, then k is odd.

We show that G_A has a w.s. s . Conditions (iii) and (iv) help to simplify the proof.

We describe an open game G^4 which has a w.s. $s^4 \in L(0^{2\#_2^1})[\#_1]$. We integrate s^4 to get the w.s. $s \in L(0^{2\#_2^1})[\#_1]$ for G_A . The game G^4 is similar to the auxiliary game G^2 . In G^2 , we initially play Borel auxiliary moves Q and $\langle \hat{q}, q \rangle$ which are determined by the Σ_1^0 set $\{q \mid \exists k R_C(q, k)\}$ (under consideration in the proof of Theorem 1.2):

$$G^2 \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q \\ \langle \hat{q}, q \rangle \end{array} \cdots$$

Similarly, the initial moves of G^4 are two pair of Borel auxiliary moves Q_i

and $\langle \hat{q}_i, q_i \rangle$:

$$G^4 \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_2 \\ \langle \hat{q}_2, q_2 \rangle \end{array} \begin{array}{c} Q_1 \\ \langle \hat{q}_1, q_1 \rangle \end{array} \cdots$$

For $i \in \{1, 2\}$, C_i determines Q_i and $\langle \hat{q}_i, q_i \rangle$. Player II must play so that

$$\text{v.)} \quad \hat{q}_2 = 1 \Rightarrow \hat{q}_1 = 1.$$

We do not allow II to play $\hat{q}_1 = 0$ if $\hat{q}_2 = 1$ since $C_1 \subseteq C_2$. Let δ be least such that $\hat{q}_\delta = 0$ or $\delta = 3$ so that the initial play of G^4 is as follows:

$$\text{If } \delta = 3, \text{ then the play of } G^4 \text{ is } \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_2 \\ \langle 1, - \rangle \end{array} \begin{array}{c} Q_1 \\ \langle 1, - \rangle \end{array} \cdots$$

$$\text{If } \delta = 2, \text{ then the play of } G^4 \text{ is } \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_2 \\ \langle 0, q_2 \rangle \end{array} \begin{array}{c} Q_1 \\ \langle 1, - \rangle \end{array} \cdots$$

$$\text{If } \delta = 1, \text{ then the play of } G^4 \text{ is } \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_2 \\ \langle 0, q_2 \rangle \end{array} \begin{array}{c} Q_1 \\ \langle 0, q_1 \rangle \end{array} \cdots$$

Let $\vec{Q} = \langle Q_i | \hat{q}_i = 1 \rangle$ and $\vec{q} = \langle q_i | \hat{q}_i = 0 \rangle$. Let μ be least such that $R_{C_\delta}(q_\delta, \mu)$ whenever $\hat{q}_i = 0$ for some i ; otherwise μ is undefined. G^4 contains a sequence $\langle (T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n \hat{t}_j = 0 \rangle$ of Borel auxiliary moves. The sequence

$$\langle (T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n \hat{t}_j = 0 \rangle$$

and the Π_1^0 set B are related via R_B , and player I may only play $T_i \in L(\vec{Q})[\#_1]$. If II plays $\hat{q}_2 = 1$, then $\vec{Q} = \langle Q_i | i = 1, 2 \rangle$, \vec{q} is the empty sequence, and the moves of G^4 following $\langle \hat{q}_1, q_1 \rangle$ constitute a play of $G^0(\vec{Q}; \vec{q})$. If II plays $\hat{q}_2 = 0$, then the moves of G^4 following $\langle \hat{q}_2, q_2 \rangle$ constitute a play of $G^3(q_2)$. Therefore, if II plays $\hat{q}_2 = 0$, then the moves of G^4 following $\langle \hat{q}_1, q_1 \rangle$ constitute a play of $G^1(\vec{Q}; \vec{q})$.

If $\hat{q}_2 = 1$ and all $\hat{t}_i = 0$, then the play of G^4 is

$$G^4 \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_2 \\ \langle 1, - \rangle \end{array} \begin{array}{c} Q_1 \\ \langle 1, - \rangle \end{array} \begin{array}{c} T_0 \\ \langle 0, t_0 \rangle \end{array} \begin{array}{c} x(0) \\ x(1) \end{array} \begin{array}{c} T_1 \\ \langle 0, t_1 \rangle \end{array} \begin{array}{c} x(2) \\ x(3) \end{array} \begin{array}{c} T_2 \\ \langle 0, t_2 \rangle \end{array} \begin{array}{c} x(4) \\ x(5) \end{array} \cdots$$

If $\hat{q}_2 = 0$ and all $\hat{t}_i = 0$, then the play of G^4 is

$$G^4 \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_2 \\ \langle 0, q_2 \rangle \end{array} \begin{array}{c} Q_1 \\ \langle \hat{q}_1, q_1 \rangle \end{array} \begin{array}{c} T_0 \\ \langle 0, t_0 \rangle \end{array} \begin{array}{c} x(0), \lambda_0 \\ x(1), \lambda_1 \end{array} \begin{array}{c} T_1 \\ \langle 0, t_1 \rangle \end{array} \begin{array}{c} x(2), \lambda_2 \\ x(3), \lambda_3 \end{array} \begin{array}{c} T_2 \\ \langle 0, t_2 \rangle \end{array} \begin{array}{c} x(4), \lambda_4 \\ x(5), \lambda_5 \end{array} \dots$$

If $\hat{q}_2 = 1$ and some $\hat{t}_n = 1$, a typical play of G^4 is

$$G^4 \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_2 \\ \langle 1, - \rangle \end{array} \begin{array}{c} Q_1 \\ \langle 1, - \rangle \end{array} \begin{array}{c} T_0 \\ \langle 0, t_0 \rangle \end{array} \begin{array}{c} x(0) \\ x(1) \end{array} \begin{array}{c} T_1 \\ \langle 0, t_1 \rangle \end{array} \begin{array}{c} x(2) \\ x(3) \end{array} \dots \\ \dots \begin{array}{c} T_{n-1} \\ \langle 0, t_{n-1} \rangle \end{array} \begin{array}{c} x(2n-2) \\ x(2n-1) \end{array} \begin{array}{c} T_n \\ \langle 1, - \rangle \end{array} \begin{array}{c} x(2n), \xi_0 \\ x(2n+1), \xi_1 \end{array} \begin{array}{c} x(2n+2), \xi_2 \\ x(2n+3), \xi_3 \end{array} \dots$$

If $\hat{q}_2 = 0$ and some $\hat{t}_n = 1$, a typical play of G^4 is

$$G^4 \begin{array}{c} \text{I} \\ \text{II} \end{array} \begin{array}{c} Q_2 \\ \langle 0, q_2 \rangle \end{array} \begin{array}{c} Q_1 \\ \langle \hat{q}_1, q_1 \rangle \end{array} \begin{array}{c} T_0 \\ \langle 0, t_0 \rangle \end{array} \begin{array}{c} x(0), \lambda_0 \\ x(1), \lambda_1 \end{array} \begin{array}{c} T_1 \\ \langle 0, t_1 \rangle \end{array} \begin{array}{c} x(2), \lambda_2 \\ x(3), \lambda_3 \end{array} \dots \\ \dots \begin{array}{c} T_{n-1} \\ \langle 0, t_{n-1} \rangle \end{array} \begin{array}{c} x(2n-2), \lambda_{2n-2} \\ x(2n-1), \lambda_{2n-1} \end{array} \begin{array}{c} T_n \\ \langle 1, - \rangle \end{array} \begin{array}{c} x(2n), \xi_0 \\ x(2n+1), \xi_1 \end{array} \begin{array}{c} x(2n+2), \xi_2 \\ x(2n+3), \xi_3 \end{array} \dots$$

Suppose II plays $\langle \hat{q}_2, q_2 \rangle = \langle 0, q_2 \rangle$ and k is least such that $R_{C_2}(q_2, k)$.

Then the moves of G^4 following $\langle 0, q_2 \rangle$ form a play of the $G^3(q_2)$ auxiliary game determined by

$$B, C_1, \langle A_\alpha | \alpha < \omega^2 \rangle, \text{ and } k \text{ (see Definition 1.2).}$$

Furthermore, ordinal auxiliary moves λ_i 's are played, and the λ_i 's are properly ordered with respect to $\bar{x}(i+1)$ and $\langle A_\alpha | \alpha < \omega \cdot (\mu + 1) \rangle$ using $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu \rangle$.

Once II plays $\hat{t}_n = 1$, ordinal auxiliary moves ξ_i 's are played, independent of what II has played for the \hat{q}_i 's, and the ξ_i 's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n + 1) \rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T_n))} | i \leq n \rangle$.

Player I wins G^4 iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. G^4 is an open game. Therefore, define for each ordinal α , P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in ON} P_\alpha$.

If p is a legal position in G^4 , let ℓ_p denote the set of legal positions in G^4 consistent with p . The set of legal positions for G^4 is in $L(0^{2\#_2^1})[\#_1]$.

Using $\langle P_\alpha | \alpha \in \text{ON} \rangle$ and Theorem 0.14, define a wellordering \prec of the legal positions for G^4 and define the canonical w.s. s^4 for G^4 so that Lemma 1.4.1 (below) and the following hold: s^4 is definable in $L(0^{2\#_2^1})[\#_1]$, and if p is a legal position in G^4 , then $s^4|_{\ell_p}$ is a w.s. for G_p^4 and is definable in any inner model of ZF in which $\prec|_{\ell_p}$ is definable. Analogous to Lemma 1.2.1, we have the following:

Lemma 1.4.1. Let p be a legal position in G^4 . Then $s^4|_{\ell_p}$ is a w.s. for G_p^4 and each of the following hold:

vi.) If p includes the move $\langle \hat{q}_1, q_1 \rangle$ and $\hat{q}_2 = 0$, then $s^4|_{\ell_p}$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu \rangle$. If p includes the move $\langle \hat{q}_1, q_1 \rangle$ and $\hat{q}_2 = 1$, then $s^4|_{\ell_p}$ is definable in $L(\vec{Q})[\#_1]$.

vii.) If p includes the move $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $s^4|_{\ell_p}$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{Q}, T_n))} | i \leq n \rangle$.

viii.) If p includes the move $\langle 0, q_2 \rangle$ and k is least such that $R_{C_2}(q_2, k)$, then $s^4|_{\ell_p}$ is definable in $L(0^{\#_2^1})[\#_1]$ from $\langle \omega_{i+1}^{L(0^{2\#_2^1})} | i \leq k \rangle$.

For legal positions p in G^4 which satisfy the hypothesis of (vi), (vii), or (viii), Properties (vi), (vii), and (viii) will make it possible to integrate $s^4|_{\ell_p}$ in such a way that we obtain a w.s. s for G_A .

Claim I: Player I has a w.s. for G_A if he has one for G^4 .

Let's first consider the case in which $\langle \cdot \rangle \in P$. Then $s^4 \in L(0^{2\#_2^1})[\#_1]$ is a w.s. for I in G^4 . We use s^4 to define a w.s. s for I in G_A . Define Q_2 and Q_1 so that the position $p_0 = (Q_2; \langle 1, - \rangle; Q_1; \langle 1, - \rangle)$ is consistent with s^4 . By Lemma 1.4.1(vi), $s^4|_{\ell_{p_0}}$ is definable in $L(\vec{Q})[\#_1]$. By Theorem 1.0, integrate $s^4|_{\ell_{p_0}}$ so as to obtain a w.s. $s_0 \in L(\vec{Q})[\#_1]$ for $A(\vec{Q})$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); \dots; x(i-1))$ such that $\forall j \in \{1, 2\} \forall i' \leq i \forall k \neg R_{C_j}(\bar{x}(i'), k)$.

Suppose we reach a position such that $\exists j \exists i \exists k R_{C_j}(\bar{x}(i), k)$. Since $C_1 \subseteq C_2$, there is a least i such that $\exists k R_{C_2}(\bar{x}(i), k)$. By (iv), i is odd. Let k be least such that $R_{C_2}(\bar{x}(i), k)$. Let $Q_2 = s^4(\cdot)$ (as above), $\langle \hat{q}_2, q_2 \rangle = \langle 0, \bar{x}(i) \rangle$, and $p_1 = (Q_2; \langle 0, q_2 \rangle)$. By Lemma 1.4.1(viii), $s^4|_{\ell_{p_1}}$ is definable in $L(0^{\#_2^1})[\#_1]$ from $\langle \omega_{i+1}^{L(2\#_2^1(0))} | i \leq k \rangle$. Since $0^{2\#_2^1}$ exists, by Corollary 1.3.1, integrate $s^4|_{\ell_{p_1}}$ so as to obtain a w.s. $s_1 \in L(0^{2\#_2^1})$ for $A(q_2)$. Let $s(p) = s_1(p)$ for any position p which extends q_2 .

Claim: The strategy s of player I is a w.s. in G_A .

Suppose x is a play of G_A consistent with s . By the definition of s , there exists a w.s. $s_0 \in L(\vec{Q})[\#_1]$ for $A(\vec{Q})$ (where $\vec{Q} = \langle Q_i \mid i \in \{1, 2\} \rangle$) such that $x(2i) = s_0(x(1); x(3); \dots; x(2i-1))$ whenever

$$\forall j \in \{1, 2\} \forall i' \leq 2i \forall n \neg R_{C_j}(\bar{x}(i'), n).$$

If $\forall j \in \{1, 2\} \forall i \forall n \neg R_{C_j}(\bar{x}(i), n)$ holds, then $x \in A(\vec{Q})$ so that $x \in A$ by Theorem 1.0.

Otherwise, $\exists j \in \{1, 2\} \exists i \exists n R_{C_j}(\bar{x}(i), n)$. Since $C_1 \subseteq C_2$, $\exists n R_{C_2}(\bar{x}(i), n)$

for some least i . By the definition of s , there exists a w.s. $s_1 \in L(0^{2\#_2^1})$ for $A(\bar{x}(i))$ such that

$$x(2k) = s_1(x(1); x(3); \dots; x(2k-1)) \text{ whenever } 2k > i.$$

Therefore, $x \in A(\bar{x}(i))$ so that $x \in A$. Thus, s is a win for I.

Claim II: Player II has a w.s. for G_A if he has one for G^4 .

Now let's consider the case $\langle \rangle \notin P$. We integrate II's w.s. s^4 for G^4 to get the w.s. $s \in L(0^{2\#_2^1})$ for II in G_A . Let

$$Q_2 = \{ \text{positions } q \text{ in } G_A \mid \forall Q' \in L(2\#_2^1(0))[\#_1] \langle 0, q \rangle \neq s^4(Q') \}.$$

Also let $Q_1 = Q_2$. Then $\langle 1, - \rangle = s^4(Q_2)$ and $Q_i \in L(0^{2\#_2^1})[\#_1]$ for $i \in \{1, 2\}$.

Let $p_0 = (Q_2; \langle 1, - \rangle; Q_1; \langle 1, - \rangle)$. By (v), p_0 is consistent with s^4 .

By Lemma 1.4.1(vii), $s^4|_{\ell_{p_0}}$ is a w.s. for $A(\vec{Q})$ (i.e. for $A(Q_2)$ since $Q_1 = Q_2$) and is definable in $L(Q_2)[\#_1]$. By Theorem 1.0, integrate $s^4|_{\ell_{p_0}}$ so as to obtain a w.s. $s_0 \in L(Q_2)[\#_1]$ for the game $A(Q_2)$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); \dots; x(i-1))$ such that $\forall i' \leq i \bar{x}(i') \in Q_2$.

Suppose we reach a position $\bar{x}(i)$ (of least length) such that $\bar{x}(i) \notin Q_2$. Then i is odd and there exists $Q'_2 \in L(0^{2\#_2^1})[\#_1]$ such that the position $p_1 = (Q'_2; \langle 0, \bar{x}(i) \rangle)$ is consistent with s^4 . By Corollary 1.3.1, obtain a w.s. $s_1 \in L(2\#_2^1(0))$ for $A(\bar{x}(i))$. Let $s(p) = s_1(p)$ for any position p which extends $\bar{x}(i)$.

Claim: The strategy s of player II is a w.s. for G_A .

Suppose x is a play of G_A consistent with s . By the definition of s , there

exists a w.s. $s_0 \in L(Q_2)[\#_1]$ for $A(Q_2)$ such that

$$x(2i+1) = s_0(x(0); x(2); \dots; x(2i)) \text{ whenever } \forall i' \leq 2i+1 \bar{x}(i') \in Q_2.$$

If $\forall i \bar{x}(i) \in Q_2$, then $x \notin A(Q_2)$ so that $x \notin A$.

Otherwise, there exists (odd) i such that $\bar{x}(i) \notin Q_2$ and $\forall j < i \bar{x}(j) \in Q_2$.

By the definition of s , there exists a w.s. $s_1 \in L(2\#_2^1(0))$ for $A(\bar{x}(i))$ such that $x(2k+1) = s_1(x(0); x(2); \dots; x(2k))$ whenever $2k+1 \geq i$. Therefore, since s_1 is a w.s. for II, $x \notin A(\bar{x}(i))$ so that $x \notin A$. Consequently, s is a w.s. in G_A of the player for whom s^4 is a w.s. ■

Now we show that the existence of indiscernibles for $L(0^{2\#_2^1})[\#_1^1]$ implies the determinacy of $(\Pi_1^0, 2 * \Sigma_1^0)_+^*$. We obtain the proof of this theorem by making changes to the proof of Theorem 1.4. These changes are analogous to the changes we made to the proof of Theorem 1.2 so as to obtain Theorem 1.3.

Theorem 1.5. If $0^{3\#_2^1}$ exists (i.e. $L(0^{2\#_2^1})[\#_1^1]$ has indiscernibles), then $\text{Det}(\Pi_1^0, 2 * \Sigma_1^0)_+^*$.

Proof: Assume $L(0^{2\#_2^1})[\#_1^1]$ has an uncountable set C_3^1 of indiscernibles. Let $B \in \Pi_1^0$, $C_1 \in \Sigma_1^0$, $C_2 \in \Sigma_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ strongly witness $A \in (\Pi_1^0, 2 * \Sigma_1^0)_+^*$. Wlog $C_1 \subseteq C_2$. Then there exist R_B , R_{C_1} , and R_{C_2} in Δ_1^0 such that

- i.) $B(x, n) \leftrightarrow \forall k R_B(\bar{x}(k), n)$;
- ii.) $C_i(x, n) \leftrightarrow \exists k R_{C_i}(\bar{x}(k), n)$;

iii.) if $\neg R_B(\bar{x}(k), n)$ and $\forall j < k R_B(\bar{x}(j), n)$, then k is odd; and

iv.) if $R_{C_i}(\bar{x}(k), n)$ and $\forall j < k \neg R_{C_i}(\bar{x}(j), n)$ for some $i \in \{1, 2\}$, then k is odd.

Conditions (iii) and (iv) help to simplify the proof.

We describe an open game G^5 which has a w.s. $s^5 \in L(0^{2\#_2^1})[\#_1]$. We complete the proof by integrating s^5 to get the w.s. $s \in L(0^{3\#_2^1})$ for G_A . The moves of G^5 are the same as the moves of G^4 with one exception: In G^5 , ordinal auxiliary moves λ_{2i} and λ_{2i+1} are respectively played with integer moves $x(2i)$ and $x(2i + 1)$ whenever $\hat{q}_2 = 1$; whereas, no ordinal auxiliary moves λ_i are played in G^4 whenever II plays $\hat{q}_2 = 1$.

If all $\hat{t}_i = 0$, then the play of G^5 is

$$\begin{array}{cccccccccccc} \text{I} & Q_2 & Q_1 & T_0 & x(0), \lambda_0 & T_1 & x(2), \lambda_2 & T_2 & x(4), \lambda_4 & \cdots \\ \text{II} & \langle \hat{q}_2, q_2 \rangle & \langle \hat{q}_1, q_1 \rangle & \langle 0, t_0 \rangle & x(1), \lambda_1 & \langle 0, t_1 \rangle & x(3), \lambda_3 & \langle 0, t_2 \rangle & x(5), \lambda_5 & \cdots \end{array}$$

If some $\hat{t}_n = 1$, the play of G^5 is

$$\begin{array}{cccccccccccc} \text{I} & Q_2 & Q_1 & T_0 & x(0), \lambda_0 & T_1 & x(2), \lambda_2 & \cdots \\ \text{II} & \langle \hat{q}_2, q_2 \rangle & \langle \hat{q}_1, q_1 \rangle & \langle 0, t_0 \rangle & x(1), \lambda_1 & \langle 0, t_1 \rangle & x(3), \lambda_3 & \cdots \\ & \cdots T_{n-1} & x(2n-2), \lambda_{2n-2} & T_n & x(2n), \xi_0 & x(2n+2), \xi_2 \\ & \langle 0, t_{n-1} \rangle & x(2n-1), \lambda_{2n-1} & \langle 1, - \rangle & x(2n+1), \xi_1 & x(2n+3), \xi_3 & \cdots \end{array}$$

In either case, if II plays $\hat{q}_2 = 1$, then II must play $\hat{q}_1 = 1$. Let $\vec{Q} = \langle Q_i \mid \hat{q}_i = 1 \rangle$. The moves of G^5 following $\langle \hat{q}_2, q_2 \rangle$ constitute a play of $G^3(q_2)$ whenever II plays $\hat{q}_2 = 0$. If II plays $\hat{q}_2 = \hat{q}_1 = 1$, then the moves of G^5 following $\langle \hat{q}_1, q_1 \rangle$ constitute a play of $G^1(\vec{Q})$. Player I must play so that $Q_2 \in L(0^{2\#_2^1})[\#_1]$, $Q_1 \in L(0^{2\#_2^1})[\#_1]$ if $\hat{q}_2 = 1$, $Q_1 \in L(0^{\#_2^1})[\#_1]$ if $\hat{q}_2 = 0$, and each $T_i \in L(\vec{Q})[\#_1]$. Moreover, the Borel auxiliary moves must satisfy the

same conditions as in G^4 : $\{q|\exists k R_{C_i}(q, k)\}$ (for $i \in \{1, 2\}$) determines the Borel auxiliary moves Q_i and $\langle \hat{q}_i, q_i \rangle$, and the sequence $\langle (T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n \hat{t}_j = 0 \rangle$ of Borel auxiliary moves and the Π_1^0 set B are related via R_B .

If there exists a least δ such that $\hat{q}_\delta = 0$, let μ be least such that $R_{C_\delta}(q, \mu)$; otherwise, let $\mu = m$. The λ_i 's are properly ordered with respect to

$$\langle A_\alpha | \alpha < \omega \cdot (\mu + 1) \rangle \text{ using } \langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq \mu \rangle.$$

(Notice $\#_2^1(\vec{Q})$ can equal $3\#_2^1(0)$.) If $\hat{t}_n = 1$, the ξ_i 's are properly ordered with respect to $\langle A_\alpha | \alpha < \omega \cdot (n + 1) \rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{Q}, T_n))} | i \leq n \rangle$.

Player I wins G^5 iff a (legal) position (of odd length) is reached at which II cannot make a (legal) move. G^5 is an open game and therefore we define, for each ordinal α , P_α as the set of positions with ordinal α and let $P = \bigcup_{\alpha \in ON} P_\alpha$. Also, if p is a legal position in G^5 , let ℓ_p denote the set of legal positions in G^5 consistent with p . The set of legal positions for G^5 is in $L(0^{2\#_2^1})[\#_1]$. Moreover, whenever p is a legal position of G^5 , the following properties hold:

- v.) If p includes the move $\langle \hat{q}_2, q_2 \rangle = \langle 1, - \rangle$, then $L(0^{2\#_2^1})[\#_1]$ contains ℓ_p .
- vi.) If p includes the move $\langle \hat{q}_2, q_2 \rangle = \langle 0, q_2 \rangle$, then $L(0^{\#_2^1})[\#_1]$ contains ℓ_p .
- vii.) If p includes the moves $\langle \hat{q}_i, q_i \rangle$ for $i = 1, 2$, then $L(\vec{Q})[\#_1]$ contains ℓ_p .
- viii.) If p includes the moves $\langle \hat{q}_i, q_i \rangle$ for $i = 1, 2$ and $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $\ell_p \in L(\vec{Q}, T_n)$ and \vec{Q} and T_n are coded by a real in $L(\vec{Q})[\#_1]$.

$L(0^{2\#_2^1})[\#_1]$ has a definable wellordering \prec of the legal positions for G^5

such that whenever p is a legal position in G^5 , $\prec | \ell_p$ is a wellordering of the legal positions of G^5 consistent with p and the following hold:

ix.) If p includes the move $\langle 0, q_2 \rangle$ and k is least such that $R_{C_2}(q_2, k)$, then $\prec | \ell_p$ is definable in $L(0\#_2^1)[\#_1]$ from $\langle \omega_{i+1}^{L(2\#_2^1(0))} | i \leq k \rangle$.

x.) Let $M = L(3\#_2^1(0))$ if $\hat{q}_2 = 1$, $M = L(2\#_2^1(0))$ if $\hat{q}_2 = 0$ and $\hat{q}_1 = 1$, and $M = L(\#_2^1(0))$ if $\hat{q}_1 = 0$. If p includes the moves $\langle \hat{q}_i, q_i \rangle$ for $i = 1, 2$, then $\prec | \ell_p$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^M | i \leq \mu \rangle$.

xi.) If p includes the move $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $\prec | \ell_p$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{Q}, T_n))} | i \leq n \rangle$.

By Lemma 0.14, use \prec to define s^5 to be the canonical w.s. for G^5 . Then s^5 is definable in $L(0^2\#_2^1)[\#_1]$. If p is a legal position in G^5 , then $s^5 | \ell_p$ is a w.s. for G_p^5 and is definable in any inner model of ZF in which $\prec | \ell_p$ is definable. Therefore, s^5 has the following properties:

Lemma 1.5.1. Let p be a legal position in G^5 . Then $s^5 | \ell_p$ is a w.s. for G_p^5 and each of the following hold:

xii.) If p includes the move $\langle 0, q_2 \rangle$ and k is least such that $R_{C_2}(q_2, k)$, then $s^5 | \ell_p$ is definable in $L(0\#_2^1)[\#_1]$ from $\langle \omega_{i+1}^{L(2\#_2^1(0))} | i \leq k \rangle$.

xiii.) If p includes the moves $\langle \hat{q}_i, q_i \rangle$ for $i = 1, 2$ and M is as in (x), then $s^5 | \ell_p$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^M | i \leq \mu \rangle$.

xiv.) If p include the move $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then $s^5 | \ell_p$ is definable in $L(\vec{Q}, T_n)$ from $\langle \omega_{i+1}^{L(\#_1(\vec{Q}, T_n))} | i \leq n \rangle$.

If p includes the move $\langle 0, q_2 \rangle$, then we use Properties (xii) and indiscernibles for $L(0^{\#_2^1})[\#_1]$ to integrate $s^5|_{\ell_p}$ with respect to the λ_i 's. If p is a legal position in G^5 which includes the move $\langle \hat{q}_1, q_1 \rangle$, we use Properties (xiii) and indiscernibles for $L(\vec{Q})[\#_1]$ to integrate $s^5|_{\ell_p}$ with respect to the λ_i 's. If p is a legal position in G^5 which includes the move $\langle \hat{t}_n, t_n \rangle = \langle 1, - \rangle$, then \vec{Q} and T_n are coded by a real in $L(\vec{Q})[\#_1]$ and we use Property (xiv) and indiscernibles for $L(\vec{Q}, T_n)$ to integrate $s^5|_{\ell_p}$ with respect to the ξ_i 's.

Claim I: Player I has a w.s. for G_A if he has one for G^5 .

Let's first consider the case in which $\langle \rangle \in P$. Then $s^5 \in L(0^{2\#_2^1})[\#_1]$ is a w.s. for I in G^5 . We use s^5 to define a w.s. s for I in G_A . Let Q_2 and Q_1 be such that $p_0 = (Q_2; \langle 1, - \rangle; Q_1; \langle 1, - \rangle)$ is consistent with s^5 . By Lemma 1.5.1(xiii), $s^5|_{\ell_{p_0}}$ is definable in $L(\vec{Q})[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(\vec{Q}))} | i \leq m \rangle$. Since $L(\vec{Q})[\#_1]$ has indiscernibles, by Theorem 1.1 obtain a w.s. $s_0 \in L(\#_2^1(\vec{Q}))$ for $A(\vec{Q})$ by integrating $s^5|_{\ell_{p_0}}$. Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); \dots; x(i-1))$ such that $\forall j \in \{1, 2\} \forall i' \leq i \forall k \neg R_{C_j}(\bar{x}(i'), k)$.

Suppose we reach a position such that $\exists j \exists i \exists k R_{C_j}(\bar{x}(i), k)$. Since $C_1 \subseteq C_2$, there is a least i such that $\exists k R_{C_2}(\bar{x}(i), k)$. By (iv), i is odd. Let k be least such that $R_{C_2}(\bar{x}(i), k)$. Let $Q_2 = s^5(\)$ (as above), $\langle \hat{q}_2, q_2 \rangle = \langle 0, \bar{x}(i) \rangle$, and $p_1 = (Q_2; \langle 0, q_2 \rangle)$. By Lemma 1.5.1(xii), $s^5|_{\ell_{p_1}}$ is definable in $L(0^{\#_2^1})[\#_1]$ from $\langle \omega_{i+1}^{L(2\#_2^1(0))} | i \leq k \rangle$. Since $L(0^{\#_2^1})[\#_1]$ has indiscernibles, by Theorem 1.3, integrate $s^5|_{\ell_{p_1}}$ so as to obtain a w.s. $s_1 \in L(0^{2\#_2^1})$ for $A(q_2)$. Let $s(p) = s_1(p)$

for any position p which extends q_2 .

Claim: The strategy s of player I is a w.s. in G_A .

Suppose x is a play of G_A consistent with s . By the definition of s , there exists a w.s. $s_0 \in L(\#_2^1(\vec{Q}))$ for $A(\vec{Q})$ (where $\vec{Q} = \langle Q_i | i = 1, 2 \rangle$) such that $x(2i) = s_0(x(1); x(3); \dots; x(2i - 1))$ whenever

$$\forall j \in \{1, 2\} \forall i' \leq 2i \forall n \neg R_{C_j}(\bar{x}(i'), n).$$

If $\forall j \in \{1, 2\} \forall i \forall n \neg R_{C_j}(\bar{x}(i), n)$ holds, then $x \in A(\vec{Q})$ so that $x \in A$ by Theorem 1.1.

Otherwise, $\exists n R_{C_2}(\bar{x}(i), n)$ for some least i since $C_1 \subseteq C_2$. By the definition of s , there exists a w.s. $s_1 \in L(0^{2\#_2^1})$ for $A(\bar{x}(i))$ such that

$$x(2k) = s_1(x(1); x(3); \dots; x(2k - 1)) \text{ whenever } 2k > i.$$

Therefore, $x \in A(\bar{x}(i))$ so that $x \in A$. Thus, s is a win for I.

Claim II: Player II has a w.s. for G_A if he has one for G^5 .

Now let's consider the case $\langle \rangle \notin P$. We integrate II's w.s. s^5 for G^5 to get the w.s. $s \in L(0^{3\#_2^1})$ for II in G_A . Let

$$Q_2 = \{\text{positions } q \text{ in } G_A \mid \forall Q' \in L(2\#_2^1(0))[\#_1] \langle 0, q \rangle \neq s^5(Q')\}.$$

Also let $Q_1 = Q_2$. Then $\langle 1, - \rangle = s^5(Q_2)$, $Q_1 = Q_2 \in L(0^{2\#_2^1})[\#_1]$, and since $\hat{q}_2 = 1$, $\langle 1, - \rangle = s^5(Q_2; \langle 1, - \rangle; Q_1)$.

Let $p_0 = (Q_2; \langle 1, - \rangle; Q_1; \langle 1, - \rangle)$. By Lemma 1.5.1(xiii), $s^5|_{\ell_{p_0}}$ is a w.s. for $A(\vec{Q})$ (i.e. for $A(Q_2)$ since $Q_1 = Q_2$) and is definable in $L(Q_2)[\#_1]$ from $\langle \omega_{i+1}^{L(\#_2^1(Q_2))} \mid i \leq m \rangle$. Since $Q_2 \in L(2\#_2^1(0))[\#_1]$ and $0^{3\#_2^1}$ exists, indiscernibles

for $L(Q_2)[\#_1]$ exist. (The previous line is the point at which this proof breaks down when one tries to prove $\text{Det}(\Pi_1^0, 2 * \Sigma_1^0)_+$ from a weaker determinacy hypothesis than the existence of $0^{3\#_2^1}$.) By Theorem 1.1, integrate $s^5|_{\ell_{p_0}}$ so as to obtain a w.s. $s_0 \in L(\#_2^1(Q_2)) \subseteq L(0^{3\#_2^1})$ for the game $A(Q_2)$. (Notice $\#_2^1(Q_2)$ can equal $0^{3\#_2^1}$.) Let $s(p) = s_0(p)$ for any position $p = (x(0); x(1); x(2); \dots; x(i-1))$ such that $\forall i' \leq i \bar{x}(i') \in Q_2$.

Suppose we reach a position $\bar{x}(i)$ (of least length) such that $\bar{x}(i) \notin Q_2$. Then i is odd and there exists $Q'_2 \in L(0^{2\#_2^1})[\#_1]$ such that the position $p_1 = (Q'_2; \langle 0, \bar{x}(i) \rangle)$ is consistent with s^5 . Let k be least such that $R_{C_2}(\bar{x}(i), k)$. By Lemma 1.5.1(xii), $s^5|_{\ell_{p_1}}$ is definable in $L(\#_2^1(0))[\#_1]$ from $\langle \omega^{L(2\#_2^1(0))} | i \leq k \rangle$. By Theorem 1.3, obtain a w.s. $s_1 \in L(2\#_2^1(0))$ for $A(\bar{x}(i))$ by appropriately integrating $s^5|_{\ell_{p_1}}$. Let $s(p) = s_1(p)$ for any position p which extends $\bar{x}(i)$.

Claim: The strategy s of player II is a w.s. for G_A .

Suppose x is a play of G_A consistent with s . By the definition of s , there exists a w.s. $s_0 \in L(\#_2^1(Q_2))$ for $A(Q_2)$ such that $x(2i+1) = s_0(x(0); x(2); \dots; x(2i))$ whenever $\forall i' \leq 2i+1 \bar{x}(i') \in Q_2$. If $\forall i \bar{x}(i) \in Q_2$, then $x \notin A(Q_2)$ so that $x \notin A$.

Otherwise, there exists (odd) i such that $\bar{x}(i) \notin Q_2$ and $\forall j < i \bar{x}(j) \in Q_2$. By the definition of s , there exists a w.s. $s_1 \in L(2\#_2^1(0))$ for $A(\bar{x}(i))$ such that $x(2k+1) = s_1(x(0); x(2); \dots; x(2k))$ whenever $2k+1 \geq i$. Therefore, since s_1 is a w.s. for II, $x \notin A(\bar{x}(i))$ so that $x \notin A$. Consequently, s is a w.s. in G_A of

the player for whom s^5 is a w.s. ■

Definition 1.3. Let $B \in \Pi_1^0$, $C_1 \in \Sigma_1^0$, $C_2 \in \Sigma_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ strongly witness $A \in (\Pi_1^0, 2 * \Sigma_1^0)_+^*$. Then we refer to the auxiliary game G^5 described in the proof of Theorem 1.5 as *the G^5 auxiliary game determined by $B \in \Pi_1^0$, $C_1 \in \Sigma_1^0$, $C_2 \in \Sigma_1^0$, $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$.*

Suppose $\vec{U} = \langle U_i \mid i < \beta \rangle$ and $\vec{u} = \langle u_i \mid i < \gamma \rangle$ respectively are a finite sequence of I-imposed subgames of G_A and a sequence of legal positions of G_A . Then *the $G^5(\vec{U}; \vec{u})$ auxiliary game determined by B , C_1 , C_2 , $\langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$* is the game which has exactly the same moves as G^5 , player I wins iff a position is reached at which II cannot make a (legal) move, and the moves of the G^5 auxiliary game are subject to the following conditions:

- i.) Each $\bar{x}(i) \in \bigcap_{i < \beta} U_i$ and each $\bar{x}(i)$ must be consistent with every u_i .
- ii.) $Q_2 \in L(2\#_2^1(\vec{U}))[\#_1]$. If $\hat{q}_2 = 1$, then $Q_1 \in L(2\#_2^1(\vec{U}))[\#_1]$. If $\hat{q}_2 = 0$, then $Q_1 \in L(\#_2^1(\vec{U}))[\#_1]$.
- iii.) $T_i \in L(\vec{U}, \vec{Q})[\#_1]$.
- iv.) Let $M = L(3\#_2^1(\vec{U}))$ and $\mu = m$ if $\hat{q}_2 = 1$. If $\hat{q}_2 = 0$ and $\hat{q}_1 = 1$, let $M = L(2\#_2^1(\vec{U}))$ and μ be least such that $R_{C_2}(q_2, \mu)$. If $\hat{q}_1 = 0$, let $M = L(\#_2^1(\vec{U}))$ and μ be least such that $R_{C_1}(q_1, \mu)$. The λ_i 's are properly ordered with respect to $\langle A_\alpha \mid \alpha < \omega \cdot (\mu + 1) \rangle$ using $\langle \omega_{i+1}^M \mid i \leq \mu \rangle$.
- v.) If $\hat{t}_n = 1$, the ξ_i 's are properly ordered with respect to $\langle A_\alpha \mid \alpha < \omega \cdot (n+1) \rangle$ using $\langle \omega_{i+1}^{L(\#_1^1(\vec{U}, \vec{Q}, T_n))} \mid i \leq n \rangle$.

vi.) $\{q|\exists k R_{C_i}(q, k)\}$ determines the Borel auxiliary moves Q_i and $\langle \hat{q}_i, q_i \rangle$.

vii.) The sequence $\langle (T_n; \langle \hat{t}_n, t_n \rangle) | \forall j < n \hat{t}_j = 0 \rangle$ of Borel auxiliary moves and the Π_1^0 set B are related via R_B .

These conditions are analogous to the conditions for the moves of G^5 . The last two are conditions which the moves of G^5 must satisfy. The others are derived by changing the conditions for the moves of G^5 so that we obtain conditions that are consistent with \vec{U} and \vec{u} . We refer to $G^5(\vec{U}; \vec{u})$ instead of the $G^5(\vec{U}; \vec{u})$ auxiliary game determined by $B, C_1, C_2, \langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ whenever $B, C_1, C_2, \langle A_\alpha \mid \alpha < \omega^2 \rangle$, and $m \in \omega$ are clear from the context. Analogous to Theorem 1.5, we have the following:

Corollary 1.5.1. Let $B, C_1, C_2, \langle A_\alpha \mid \alpha < \omega^2 \rangle, m, A, \vec{U}$, and \vec{u} be as in Definition 1.3. Let p be a legal position of a game G^* such that the moves of G^* following p constitute a play of $G^5(\vec{U}; \vec{u})$. Suppose there exists a definable wellordering for \vec{U} in $L(\vec{U})$, $3\#_2^1(\vec{U})$ exists, and s^* is a w.s. for G^* such that $s^*|_{\ell_p} \in L(2\#_2^1(\vec{U}))[\#_1]$. Then $s^*|_{\ell_p}$ can be integrated so as to obtain a w.s. $s_p \in L(3\#_2^1(\vec{U}))$ for $A(\vec{U}; \vec{u})$ such that the following hold:

- i.) s_p is a w.s. of the player for whom s^* is a w.s.,
- ii.) If s^* is a w.s. for I, \hat{p} is a position consistent with s_p , and the moves in \hat{p} of player II are consistent with \vec{u} , then $\hat{p} \in \bigcap_{i < \beta} U_i$. Therefore, if s^* is a w.s. for I and x is a play consistent with s_p , then $x \in A(\vec{u})$.

- iii.) Let \hat{p} be a position consistent with s_p and with \vec{U} . If the moves in \hat{p} of

the player for whom s_p is not a w.s. are consistent with \vec{u} , then \hat{p} is consistent with \vec{u} . ■

Next we prove generalizations of Theorem 1.4 and Theorem 1.5. In Theorem 1.6, we show that if $L(0^{\beta\#_2^1})[\#_1] \models$ “every real has a sharp,” then every $(\Pi_1^0, \beta * \Sigma_1^0)^*$ game has a w.s. in $L(0^{\beta\#_2^1})[\#_1]$. In Theorem 1.7, we show that if $L(0^{\beta\#_2^1})[\#_1]$ has indiscernibles, then every $(\Pi_1^0, \beta * \Sigma_1^0)_+^*$ game has a w.s. in $L(0^{(\beta+1)\#_2^1})$. Already we have shown Theorems 1.6 and 1.7 for $\beta \leq 2$. The proof of Theorems 1.6 and 1.7 is by induction on β . Assume the following as our induction hypothesis to the proofs of Theorems 1.6 and 1.7:

Induction Hypothesis. Suppose $0^{\beta\#_2^1}$ exists and

$$B \in \Pi_1^0, C_1, C_2, C_3, \dots, C_{\beta-1} \in \Sigma_1^0, \langle A_\alpha \mid \alpha < \omega^2 \rangle, \text{ and } m \in \omega$$

witness that A is $(\Pi_1^0, (\beta - 1) * \Sigma_1^0)_+^*$. Then A has a w.s. $s \in L(0^{\beta\#_2^1})$. If $s^{2\beta-1}$ is a w.s. for the $G^{2\beta-1}$ auxiliary game determined by

$$B, C_1, C_2, C_3, \dots, C_{\beta-1}, \langle A_\alpha \mid \alpha < \omega^2 \rangle, \text{ and } m \in \omega,$$

then s is a w.s. of the player for whom $s^{\beta-1}$ is. (See Definition 1.4, which follows Theorem 1.7, for the definition of the $G^{2\beta-1}$ auxiliary game.)