§1. **Preliminaries**

In this section, we introduce the notation, the inner models, and the relevant classes. We also introduce some terminology for the standard set-up involving Borel and ordinal auxiliary moves. This terminology will help us avoid repetition in Sections 2 and 3. After fixing the notation, the other topics of this section are presented in four subsections.

We use the following notation: $\omega = \{0,1,2,\cdots\}$, $\mathbf{N} = \{1,2,3,\cdots\}$, ω_i is the i^{th} uncountable cardinal in V, and ON denotes the class of ordinals. If z is a function whose domain contains $\{0,1,2,\cdots,n-1\}$, $\bar{z}(n)$ denotes the finite sequence $(z(0), z(1), z(2), \cdots, z(n-1))$. If $z(i) \in \omega$ for $i < \omega$ and p_i denotes the i^{th} prime with $p_0 = 2$, we sometimes let $\bar{z}(n)$ denote $p_0^{z(0)} p_1^{z(1)} p_2^{z(2)} \cdots p_{n-1}^{z(n-1)}$; if $u = p_0^{z(0)} p_1^{z(1)} p_2^{z(2)} \cdots p_{n-1}^{z(n-1)}$, let $(u)_i = z(i)$ and let $\ell h(u) = \mathit{length}(u)$ be the least $i \in \omega$ such that $z(j) = 0$ for $i \le j < n$. If $z \in {}^{\omega}\omega$ and $i \in \omega$, define $(z)_i \in {}^{\omega}\omega$ by $(z)_i(n) = (z(n))_i$. We also use the interval notation (but) for ordinals, e.g. $\alpha \in [\beta, \gamma) \iff \beta \le \alpha < \gamma$. We use $\frown$ for concatenation: If $\vec{u} = \langle u_i | i < \beta \rangle$ and $\vec{v} = \langle v_i | i < \gamma \rangle$, then $\vec{u} {}^\frown \vec{v} =_{df} \langle w_i | i < \beta + \gamma \rangle$, where $w_i = u_i$ for $i < \beta$ and $w_{\beta+i} = v_i$ for $i < \gamma$.

We use the least operator μ, e.g. $\mu\alpha(\cdots\alpha\cdots)$ denotes "the least α such that $\cdots\alpha\cdots$". We write $\exists^{\mu} i \in \omega(\cdots i \cdots)$ for $\exists i \in \omega[(\cdots i \cdots) \wedge \forall j < i \neg(\cdots j \cdots)]$, and additionally drop the $\in \omega$ when it is clear from the context. For $B \subseteq ({}^{\omega}\omega) \times \omega$, we define B^{μ} by: $B^{\mu}(x,n) \Leftrightarrow_{df} B(x,n) \wedge \forall j < n \neg B(x,j)$. For $R \subseteq \omega$, we define R^{μ} by: $R^{\mu}(u) \Leftrightarrow_{df} R(u) \wedge \forall i < \ell h(u) \neg R\left(p_0^{(u)_0} p_1^{(u)_1} \cdots p_i^{(u)_i}\right)$. So leastness for $B^{\mu}(x,n)$ is in terms of selecting an n corresponding to B and x, whereas leastness for $R^{\mu}(u)$ is in terms of $\ell h(u)$ in the sense that $R^{\mu}(u)$ holds requires that R^{μ} cannot hold of any proper initial segment of u.

ω_1^{CK} is the usual Church-Kleene ω_1, and it is well-known that $\beta < \omega_1^{CK}$ iff β is a recursive ordinal [see page 195, Mo80]. Also, $\beta < \omega_1^{CK}(x)$ iff β is a recursive ordinal in x (see page 246, [Mo80], where [Mo80] uses the notation ω_1^x instead of $\omega_1^{CK}(x)$). If T is a game tree and $C \subseteq [T] \times {}^{\omega}\omega$, then $[T] =_{df} \{x | \forall n\ x\lceil n \in T\}$ and

$$pC =_{df} \{x \in [T] | \exists y \in {}^{\omega}\omega\ (x,y) \in C\}.$$

We use ▪ [respectively △] to indicate the end of a proof or theorem [respectively definition]. $G(A)$ denotes the game with payoff set A. Det(Γ) and DetΓ each signify that for every $A \in \Gamma$, the game with payoff set A is determined.

§1.1. **The Inner Models**

Next we construct the inner models used in this paper. We assume the reader is familiar with a standard development of the notion of *indiscernibles* for a transitive set or class, including the case of indiscernibility over a set of parameters, such as presented in [Je78] and [Ma∞].

Definition 1.1.

(1) Let $\wp^0(\omega) = \omega$, $\wp^{k+1}(\omega)$ be the power set of $\wp^k(\omega)$, and if k is a limit ordinal, $\wp^k(\omega) = \bigcup_{i<k} \wp^i(\omega)$. Let $trcl(A)$ denote the transitive closure of $\{A\}$; that is, the smallest transitive set y such that $A \in y$. Also, let $Def(M)$ be the set of all $y \subseteq M$ such that, for some formula φ and $x_0, x_1, x_2, \ldots, x_{n-1} \in M$,

$$y = \{x \in M | (M, \in) \models \varphi[x_0, x_1, x_2, \cdots, x_{n-1}, x]\}.$$

Define $L_\xi(A)$ by induction as follows: $L_0(A) = trcl(A)$,

$$L_\xi(A) = \bigcup_{\eta<\xi} L_\eta(A) \text{ if } \xi \text{ is a limit ordinal, and}$$
$$L_{\xi+1}(A) = Def(L_\xi(A)).$$

Finally let $L(A) = \bigcup_{\xi \in ON} L_\xi(A)$.

(2) $A^\#$ exists iff there exists a class C of indiscernibles for $L(A)$ over $trcl(A)$ which contains all uncountable cardinals and is closed.

(3) Suppose $A^\#$ exists, and let C be as in (2). $A^\#$ is defined to be the set of all Gödel numbers of formulas $\varphi(v_0, v_1, v_2, \cdots, v_n)$ for which $L(A) \models \varphi[A, \xi_0, \xi_1, \xi_2, \cdots, \xi_{n-1}]$ for some (any) increasing sequence $\xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{n-1}$ from C. We call $A^\#$ the sharp of A.

(4) Let $\#^\infty$ be the function with domain $\{A | A^\# \text{ exists}\}$ and defined by $A \mapsto A^\#$. For k an ordinal, let $\#^k$ be the restriction of $\#^\infty$ to objects of type k (so that $\#^k$ is the sharp function on $\wp^k(\omega)$ and $\#^1$ is the sharp function on the reals). △

Definition 1.2. $L(x)\left[\vec{\#}\right]$. Let $\vec{\#} = \langle \#_i | i \in I \rangle$ be a sequence of sharp functions. By transfinite recursion, we define: $L_0(x)\left[\vec{\#}\right] =_{df} trcl(x)$,

$$L_\xi(x)\left[\vec{\#}\right] =_{df} \bigcup_{\eta<\xi} L_\eta(x)\left[\vec{\#}\right] \text{ if } \xi \text{ is a limit ordinal, and}$$
$$L_{\xi+1}(x)\left[\vec{\#}\right] =_{df} \mathrm{Def}\Big(L_\xi(x)\left[\vec{\#}\right] \cup \Big\{A^{\#_i} | i \in I \text{ and } A \in L_\xi(x)\left[\vec{\#}\right] \cap \text{ dom } (\#_i)\Big\}\Big).$$

Finally we let $L(x)\left[\vec{\#}\right] = \bigcup_{\xi \in ON} L_\xi(x)\left[\vec{\#}\right]$. We define $L\left[\vec{\#}\right] =_{df} L(\emptyset)\left[\vec{\#}\right]$. △

Given a sequence $\vec{\#} = \langle \#_i | i \in I \rangle$ of sharp functions, we define the sharp function $\mathcal{S}_\infty\left(\vec{\#}\right)$ so that $\left(\mathcal{S}_\infty\left(\vec{\#}\right)\right)(x)$ codes indiscernibles for $L(x)\left[\vec{\#}\right]$.

Definition 1.3. $\mathcal{S}_k^{\gamma+1}$. Let $\vec{\#}$ be a sequence of sharp functions.

(1) $\left(\mathcal{S}_\infty\left(\vec{\#}\right)\right)(x) = x^{\mathcal{S}_\infty(\vec{\#})}$ exists iff there exists a class C of indiscernibles for $L(x)\left[\vec{\#}\right]$ over $trcl(x) \cup \{\#_i | i \in I\}$ which contains all uncountable cardinals and is closed.

(2) Suppose $\left(\mathcal{S}_\infty\left(\vec{\#}\right)\right)(x)$ exists, and let C be as in (1). $\left(\mathcal{S}_\infty\left(\vec{\#}\right)\right)(x)$ is defined to be the set of $(m, \vec{a})$ such that m is the Gödel number of a formula $\varphi(v_0, v_1, v_2, \cdots, v_{n+1})$ and $\vec{a}$ is a sequence from $trcl(x)$ for which

$$L(x)\left[\vec{\#}\right] \models \varphi\left[\vec{\#}, \vec{a}, \xi_0, \xi_1, \xi_2, \cdots, \xi_{n-1}\right]$$

for some (any) increasing sequence $\xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{n-1}$ from C.

(3) For k an ordinal, let $\mathcal{S}_k\left(\vec{\#}\right)$ be the restriction of the sharp function $\mathcal{S}_\infty\left(\vec{\#}\right)$ to objects of type k.

(4) Let $\mathcal{S}$ be either $\mathcal{S}_\infty$ or $\mathcal{S}_k$ for some ordinal k. By transfinite recursion on γ, we define the sharp function $\mathcal{S}^{\gamma+1}\left(\vec{\#}\right)$ and the sequence $\vec{\mathcal{S}}^\gamma\left(\vec{\#}\right)$ of sharp functions as follows: $\vec{\mathcal{S}}^0\left(\vec{\#}\right) =_{df} \vec{\#}$, $\mathcal{S}^1\left(\vec{\#}\right) =_{df} \mathcal{S}\left(\vec{\#}\right)$, $\vec{\mathcal{S}}^\gamma\left(\vec{\#}\right) =_{df} \vec{\#}^\frown \left\langle \mathcal{S}^{\beta+1}\left(\vec{\#}\right) | \beta < \gamma \right\rangle$, and $\mathcal{S}^{\gamma+1}\left(\vec{\#}\right) =_{df} \mathcal{S}\left(\vec{\mathcal{S}}^\gamma\left(\vec{\#}\right)\right)$. (We do not define $\mathcal{S}^\gamma\left(\vec{\#}\right)$ for γ a limit ordinal.)

(5) Let $\vec{\#}^1_\gamma =_{df} \vec{\mathcal{S}}^\gamma_1(\emptyset)$ and $\#^1_{\gamma+1} = \mathcal{S}^{\gamma+1}_1(\emptyset) = \mathcal{S}_1\left(\vec{\#}^1_\gamma\right)$. △

Note that $\#^k = \mathcal{S}_k(\emptyset)$, $\#^\infty = \mathcal{S}_\infty(\emptyset)$, $\vec{\#}^1_\gamma = \left\langle \#^1_{\beta+1} | \beta < \gamma \right\rangle$, $\vec{\#}^1_\gamma$ is a sequence of sharp functions on the reals, and $\vec{\#}^{\gamma\infty} = \langle \#^{(\beta+1)\infty} | \beta < \gamma \rangle$. $\#^1_2(r)$ exists iff $L(r)\left[\#^1\right]$ has indiscernibles. More generally, $\#^1_{\gamma+1}(r)$ exists iff $L(r)\left[\vec{\#}^1_\gamma\right]$ has indiscernibles, in which case $\#^1_{\gamma+1}(r)$ codes indiscernibles for $L(r)\left[\vec{\#}^1_\gamma\right]$.

$\#^{2\infty}(x)$ exists iff $L(x)\left[\#^\infty\right]$ has indiscernibles. $\#^{(\gamma+1)\infty}(x)$ exists iff $L(x)\left[\vec{\#}^{\gamma\infty}\right]$ has indiscernibles, in which case $\#^{(\gamma+1)\infty}(x)$ codes indiscernibles for $L(x)\left[\vec{\#}^{\gamma\infty}\right]$.

The sequences $\vec{\#}$ of sharp functions which we shall be interested here have the form:

$(*)$ $\vec{\#} = \vec{\mathcal{S}}^{\gamma(n)}_{k(n)} \vec{\mathcal{S}}^{\gamma(n-1)}_{k(n-1)} \cdots \vec{\mathcal{S}}^{\gamma(1)}_{k(1)} \vec{\mathcal{S}}^{\gamma(0)}_\infty(\emptyset) = \vec{\mathcal{S}}^{\gamma(n)}_{k(n)} \vec{\mathcal{S}}^{\gamma(n-1)}_{k(n-1)} \cdots \vec{\mathcal{S}}^{\gamma(1)}_{k(1)}\left(\vec{\#}^{\gamma(0)\infty}\right)$,

for some $\gamma(i) \in \left[0, \omega_1^{CK}(r)\right)$ and $\omega_1^{CK}(r) > k(1) > k(2) > k(3) > \cdots > k(n)$, where r is some fixed real,

(i) $\vec{\#} = \vec{\mathcal{S}}^{\gamma(0)}_\infty(\emptyset) = \vec{\#}^{\gamma(0)\infty}$ when $n = 0$, and (ii) $\vec{\mathcal{S}}^0_\infty(\emptyset) = \vec{\#}^{0\infty} = \emptyset$.

We only concern ourselves with the case $k(1) > k(2) > k(3) > \cdots > k(n)$ because $\mathcal{S}_k\left(\mathcal{S}_\ell\left(\vec{\#}\right)\right) = \mathcal{S}_k\left(\vec{\#}\right)$ for $k > \ell$ and $\vec{\#}$ as in $(*)$.

In Definition 1.3, we set up the notation $\mathcal{S}^{\gamma+1}\left(\vec{\#}\right)$ for taking $\gamma + 1$ times, $\mathcal{S}$ of $\vec{\#}$. Next we introduce the notation $(\beta\#)(x)$ for taking β times, the sharp $\#$ of an appropriate object x.

Definition 1.4. Let $\#$ be a sharp function. Let $0\#$ be the identity function so that $(0\#)(x) = x$. Let $1\# =_{df} \#$. If $(\beta\#)(x)$ has been defined and exists, and if $\#((\beta\#)(x))$ exists, define $(\beta+1)\#(x) =_{df} \#(\beta\#(x))$. If β is a limit ordinal, $(\alpha\#)(x)$ exists for all $\alpha < \beta$, and there exists a $<_{L(\langle \alpha\#(x) | \alpha < \beta \rangle)}$-least real r such that $L(r) = L(\langle \alpha\#(x) | \alpha < \beta \rangle)$, then we define $(\beta\#)(x)$ to be $\#(r)$. △

The models in which we are interested, in [Du92a,92b,95,5,6,7] and this paper, all have the form $L\left(\beta\mathcal{S}\left(\vec{\#}\right)(x)\right)\left[\vec{\#}\right]$, where for some fixed real r: $\beta < \omega_1^{CK}(r)$, $\mathcal{S} \in \left\{\mathcal{S}_\infty, \mathcal{S}_k | k < \omega_1^{CK}(r)\right\}$, $\vec{\#}$ is as in $(*)$, and $x \in \operatorname{dom}\left(\beta\mathcal{S}\left(\vec{\#}\right)\right)$. In this paper, we are interested in these models for the case $\mathcal{S} = \mathcal{S}_1$ and x is a real $\hat{r}$, so the models have the form $L\left(\left(\beta\mathcal{S}_1\left(\vec{\#}\right)\right)(\hat{r})\right)\left[\vec{\#}\right]$. In [Du5,92b], we only consider the models $L\left(\beta\#^1_{\gamma+1}(0)\right)\left[\vec{\#}^1_\gamma\right]$ for $\beta < \omega$, and this example the reader may wish to keep in mind. $L\left((\beta+1)\mathcal{S}\left(\vec{\#}\right)(\hat{r})\right)\left[\vec{\#}\right]$ properly extends $L\left(\beta\mathcal{S}\left(\vec{\#}\right)(\hat{r})\right)\left[\vec{\#}\right]$, containing as an element the real $\left((\beta+1)\mathcal{S}\left(\vec{\#}\right)\right)(\hat{r})$ which codes indiscernibles for $L\left(\beta\mathcal{S}\left(\vec{\#}\right)(\hat{r})\right)\left[\vec{\#}\right]$. If $(\beta+1)\mathcal{S}\left(\vec{\#}\right)(\hat{r})$ exists, then $L\left(\beta\mathcal{S}\left(\vec{\#}\right)(\hat{r})\right)\left[\vec{\#}\right] \models$ "each $\#$ in $\vec{\#}$ is total."

Lemma 1.5. Let r be a real, $\vec{\#}$ be as in $(*)$, and $M =_{df} L\left(\beta\mathcal{S}_1\left(\vec{\#}\right)(r)\right)\left[\vec{\#}\right]$. If $(\beta+1)\,\mathcal{S}_1\left(\vec{\#}\right)(r)$ exists, then for each $\#$ in $\vec{\#}$:

(i) $M \models$ "$\#$ is total."

(ii) $\#\lceil M = \{(x,y)\,|\,x,y \in M$ and $M \models$ "$y = \#(x)$"$\}$.

Proof: Assume $M =_{df} L\left(\beta\mathcal{S}_1\left(\vec{\#}\right)(r)\right)\left[\vec{\#}\right]$ has an uncountable set C of indiscernibles. Fix $\vec{\#}$ as in $(*)$. If $\gamma = \gamma(0)+\gamma(1)+\cdots+\gamma(i)+j$ for some $i<n$ and $j<\gamma(i+1)$, let

$$\#_{\gamma+1} =_{df} \mathcal{S}_{k(i+1)}\left(\vec{\mathcal{S}}^{j}_{k(i+1)}\vec{\mathcal{S}}^{\gamma(i)}_{k(i)}\cdots\vec{\mathcal{S}}^{\gamma(1)}_{k(1)}\vec{\mathcal{S}}^{\gamma(0)}_{\infty}(\emptyset)\right).$$

Let $\vec{\#}_\gamma = \langle \#_{\beta+1}|\beta<\gamma\rangle$ for $\gamma \le \gamma(0)+\gamma(1)+\cdots+\gamma(n)$. Then $\vec{\#} = \vec{\#}_{\gamma(0)+\gamma(1)+\cdots+\gamma(n)}$.

By induction on γ, we show (i) and (ii) for $\# = \#_{\gamma+1}$. Fix γ. Assume that (i) and (ii) hold for $\# = \#_{\xi+1}$ whenever $\xi<\gamma$. Suppose $M \models$ "$\#_{\gamma+1}$ is not total" and let $x \in M$ be $<_M$-least witnessing this. Then x is definable in M so that, since (ii) holds for $\# = \#_{\xi+1}$ whenever $\xi<\gamma$, C is a set of indiscernibles for $L(x)[\vec{\#}_\gamma]$. Hence $\#_{\gamma+1}(x)$ does exist, giving us a contradiction. Thus, (i) holds for $\# = \#_{\gamma+1}$.

To show (ii), notice that by the construction of M in Definition 1.2, $\#_{\gamma+1}(x) \in M$ for every $x \in M \cap dom(\#_{\gamma+1})$, i.e. for every $x \in M$ such that $\#_{\gamma+1}(x)$ exists.

■

§1.2. **The Relevant Classes**

In this subsection, we define the lightface classes $(\Gamma)^*$ and $(\Gamma)^*_+$ and their boldface analogues $(\Gamma)^{**}$ and $(\Gamma)^{**}_+$. We establish (in this paper) determinacy results for the lightface classes $(\Gamma)^*$ and $(\Gamma)^*_+$ for certain Γ. Boldface analogues of our results also hold, but our interests here in defining the boldface classes are so that we may state a particular result, called (HYP) in Section 3, which we use in conjunction with proofs in Sections 2 and 3 to produce additional determinacy correspondences. To properly state (HYP), we need to present codes for the boldface classes. (HYP) is used to provide an appropriate analogue of the base step for the results of Section 3 on the trees we encounter; specifically, it is used in Remark 3.4 to verify that winning strategies for the auxiliary games G^γ and G^γ_+ exist in the appropriate models. The reader willing to accept such an analogue or the existence of such winning strategies can for the most part ignore the definitions for the boldface classes and their codes. The reader does need to be familiar with the lightface classes presented here.

Definition 1.6. Let T be a game tree and $A \subseteq [T]$. Then $A \in \mathbf{\Sigma}^1_1$ iff there is a closed $C \subseteq [T] \times {}^\omega\omega$ such that $A = pC$. $A \in \mathbf{\Pi}^1_1$ iff $[T] \setminus A \in \mathbf{\Sigma}^1_1$. $\mathbf{\Delta}^1_1 = \mathbf{\Sigma}^1_1 \cap \mathbf{\Pi}^1_1$. △

Following standard practice, we abuse terminology by calling $S \subseteq T \times ({}^{<\omega}\omega)$ a tree on $[T] \times ({}^\omega\omega)$ if the following hold:

(i) $\forall \langle p,u\rangle \in S\ \ell h(p) = \ell h(u)$.

(ii) (Coordinate-wise closure under initial segments.) If $\langle p,u\rangle \in S$, $q \subseteq p$, $v \subseteq u$, and $\ell h(q) = \ell h(v)$, then $\langle q,v\rangle \in S$.

In this case, we also abuse notation, letting

$$p[S] =_{df} \{x \in [T] \,|\, \exists y \in {}^{\omega}\omega \forall n (x\lceil n, y\lceil n) \in S\}.$$

The $\mathbf{\Sigma^1_1}$ subsets of $[T]$ are the sets $p[S]$, where S is a tree on $[T] \times ({}^{\omega}\omega)$. We call (T,S) a *code* of the $\mathbf{\Pi^1_1}$ set $[T] \setminus p[S]$.

Definition 1.7. Let T be a game tree.

(i) The Difference Kernel. Let $\vec{E}$ be a sequence $\langle E_\alpha | \alpha < \beta\rangle$ of subsets of $[T]$. We denote the *difference kernel* of $\vec{E}$ by $dk\left(\vec{E}\right)$, i.e.

$$x \in dk\left(\vec{E}\right) \Leftrightarrow_{df} \mu\alpha\,(x \notin E_\alpha \text{ or } \alpha = \beta) \text{ is odd.}$$

(ii) For $\beta \in ON$, the $\beta - \mathbf{\Pi}^1_1$ subsets of $[T]$ are the difference kernels $dk\left(\langle E_\alpha|\alpha < \beta\rangle\right)$ in which each E_α is $\mathbf{\Pi}^1_1$. △

The notion of difference kernel is discussed in [Hd44]. If $\vec{E} = \langle E_\alpha|\alpha < \beta\rangle$ and $\gamma \le \beta$, we let $\vec{E}_\gamma =_{df} \langle E_\alpha|\alpha < \gamma\rangle$.

Definition 1.8. Codes. Let T be a game tree. Suppose $\beta < \omega_1^{CK}(x)$ for some real x. We call $\vec{E} = \langle E_\alpha|\alpha < \beta\rangle$ a $\mathbf{\Pi}^1_1$ *sequence* if $\left\{(z,n) \in [T] \times F | z \in E_{|n|}\right\} \in \mathbf{\Pi}^1_1$, where $|k|$ denotes the order type of $k \in \omega$ in some (any) wellordering, recursive in x, of a subset F of ω with order type β. In this case, (x,T,S) is a *code* for the $\mathbf{\Pi^1_1}$ sequence $\vec{E}$ if $(T \times \omega, S)$ is a code for the $\mathbf{\Pi^1_1}$ set $\left\{(z,n) \in [T] \times F | z \in E_{|n|}\right\}$. (So S is a tree on $[T] \times \omega \times ({}^{\omega}\omega)$.) In this case, we also call (x,T,S) a code for the $\beta - \mathbf{\Pi^1_1}$ set $dk\left(\vec{E}\right)$. When $\omega_1^{CK}(x) = \omega_1^{CK}$, we shall drop the x in (x,T,S).

Instead of defining a single code for a fixed $\beta - \mathbf{\Pi^1_1}$ set $dk\left(\langle E_\alpha|\alpha < \beta\rangle\right)$, it is probably more standard to consider a sequence $\langle c_\alpha|\alpha < \beta\rangle$ of codes corresponding to $dk\left(\langle E_\alpha|\alpha < \beta\rangle\right)$ in which c_α is a code for E_α (for $\alpha < \beta$) (see e.g. [Ma∞]). Also, since every $\mathbf{\Pi}^1_1$ set is $2 - \mathbf{\Pi}^1_1$, we have two different definitions of a code for a $\mathbf{\Pi}^1_1$ set, but no confusion arises due to consideration of the context.

We next recall the lightface analogue of $\beta - \mathbf{\Pi}^1_1$ for subsets of ${}^{\omega}\omega = [{}^{<\omega}\omega]$.

Definition 1.9. $(\beta - \Pi^1_1)(x)$. Let $\beta < \omega_1^{CK}(x)$.

(i) Let $\vec{E} = \langle E_\alpha|\alpha < \beta\rangle$, where each $E_\alpha \subseteq {}^{\omega}\omega$. $\vec{E}$ is a $\Pi^1_1(x)$ sequence if

$$\left\{(y,k) \in {}^{\omega}\omega \times F | y \in A_{|k|}\right\} \in \Pi^1_1(x),$$

where $|k|$ denotes the order type of $k \in \omega$ in some (any) wellordering recursive in x, of a subset F of ω with order type β.

(ii) A is lightface $\left(\beta - \Pi^1_1\right)(x)$ iff A is the difference kernel of a $\Pi^1_1(x)$ sequence $\vec{A} = \langle A_\alpha | \alpha < \beta\rangle$. In this case, we say that $\vec{A}$ [respectively, $\langle A_\alpha|\alpha \leq \beta\rangle$, where $A_\beta = \emptyset$] *witnesses* that A is $\left(\beta - \Pi^1_1\right)(x)$.

(iii) $\beta - \Pi^1_1$ is the class $\left(\beta - \Pi^1_1\right)(\emptyset)$, $\left(< \beta - \Pi^1_1\right) = \bigcup_{\alpha<\beta} \left(\alpha - \Pi^1_1\right)$, and $\left(< \beta - \Pi^1_1\right)(x) = \bigcup_{\alpha<\beta} \left(\alpha - \Pi^1_1\right)(x)$.

(iv) A is $\Delta(\Gamma)$ iff both A and its complement are in Γ. $\triangle$

We shall abbreviate $(\beta - \Pi^1_1)(x)$ by $\beta - \Pi^1_1(x)$. By replacing each A_α in Definition 1.9(ii) by $A'_\alpha =_{df} \bigcap_{\gamma\leq\alpha} A_\gamma$ for $\alpha < \beta$, we see that, wlog, one may also require in the definition of $\beta - \Pi^1_1(x)$ (see Definition 1.9(ii)) that $A_\gamma \supseteq A_\delta$ whenever $\gamma < \delta < \beta$. So we have the following:

Proposition 1.10. Let $\beta < \omega_1^{CK}(x)$. A is $\beta{-}\Pi^1_1(x)$ iff there exists $\langle A_\alpha|\alpha \leq \beta\rangle$ which witnesses $A \in \beta{-}\Pi^1_1(x)$ and such that

$$A_\gamma \supseteq A_\alpha \text{ whenever } \gamma \leq \alpha \leq \beta;$$

in this case, we say that $\langle A_\alpha|\alpha < \beta\rangle$ *strongly witnesses* that A is $\beta - \Pi^1_1$. ■

Proposition 1.11. The following are equivalent:

(i) A is $\Delta\left(\omega^2 - \Pi^1_1(x)\right)$.

(ii) There exists $\vec{A} = \langle A_\alpha|\alpha < \omega^2\rangle$ which witnesses $A \in \omega^2 - \Pi^1_1(x)$ and such that $\bigcap_{\alpha<\omega^2} A_\alpha = \emptyset$. In this case, we say $\vec{A}$ *witnesses* that A is $\Delta\left(\omega^2 - \Pi^1_1(x)\right)$.

(iii) There exists $\vec{A} = \langle A_\alpha|\alpha < \omega^2\rangle$ which satisfies the conditions given in (ii) and such that $A_\gamma \supseteq A_\alpha$ whenever $\gamma \leq \alpha \leq \omega^2$. In this case, we say that $\vec{A}$ *strongly witnesses* that A is $\Delta\left(\omega^2 - \Pi^1_1(x)\right)$.

Proof: That (ii) and (iii) are equivalent follow from the proof of Proposition 1.10. So we only show that (i) and (ii) are equivalent.

First assume $\bigcap_{\alpha<\omega^2} A_\alpha = \emptyset$ and let $A'_{\omega\cdot n} = {}^\omega\omega$ for $n \in \omega$ and $A'_{\alpha+1} = A_\alpha$. Then

$$x \notin A \leftrightarrow \mu\alpha(x \notin A_\alpha) \text{ is even} \leftrightarrow \mu\alpha(x \notin A'_\alpha) \text{ is odd.}$$

Thus, ${}^\omega\omega \setminus A$ is $\omega^2 - \Pi^1_1(x)$ and A is $\Delta\left(\omega^2 - \Pi^1_1(x)\right)$.

Now assume A is $\Delta\left(\omega^2 - \Pi^1_1(x)\right)$. Then $\langle B_\alpha|\alpha \leq \omega^2\rangle$ and $\langle C_\alpha|\alpha \leq \omega^2\rangle$ exist such that $B_{\omega^2} = C_{\omega^2} = \emptyset$, both $\{(k,x) \in \omega \times ({}^\omega\omega)|x \in B_{|k|}\}$ and $\{(k,x) \in \omega \times ({}^\omega\omega)|x \in C_{|k|}\}$ are $\Pi^1_1(x)$, and

$$\begin{aligned} x \in A &\leftrightarrow \exists \textit{ odd } \alpha \leq \omega^2 \textit{ such that } x \in (\bigcap_{\gamma<\alpha} B_\gamma) \setminus B_\alpha \\ &\leftrightarrow \exists \textit{ even } \alpha \leq \omega^2 \textit{ such that } x \in (\bigcap_{\gamma<\alpha} C_\gamma) \setminus C_\alpha \end{aligned}$$

Let $A_\alpha = (\bigcap_{\gamma\leq\alpha} B_\gamma) \cap (\bigcap_{\gamma<\alpha} C_\gamma)$. If $x \in \bigcap_{\alpha<\omega^2} A_\alpha$, then

$$x \in (\bigcap_{\gamma<\omega^2} B_\gamma) \setminus B_{\omega^2} \subseteq {}^\omega\omega \setminus A \text{ and } x \in (\bigcap_{\gamma<\omega^2} C_\gamma) \setminus C_{\omega^2}) \subseteq A;$$

therefore, $\bigcap_{\alpha<\omega^2} A_\alpha = \emptyset$.

If $\alpha < \omega^2$ is least such that $x \notin A_\alpha$, then either $x \notin B_\alpha$ or ($\alpha = \beta + 1$ and $x \notin C_\beta$). In either case, $x \in A$ iff α is odd. Thus,

$$A = \{x \in {}^\omega\omega | \exists \textit{ odd } \alpha \leq \beta \textit{ such that } x \in \bigcap_{\gamma<\alpha} A_\gamma \setminus A_\alpha\} = dk(\langle A_\alpha | \alpha < \omega^2 \rangle) . \qquad \blacksquare$$

We are interested in classes of sets near the "bottom" of $\Delta\left(\omega^2 - \Pi^1_1\right)$. For the remainder of this paper, we reserve the notation $\vec{A}$ to denote an ω^2 sequence of sets of $[T]$, where T will be some game tree. Define $n(x)$ to be the least $n \in \mathbf{N}$ such that $\mu\alpha\,(x \notin A_\alpha)$ exists and is $< \omega \cdot n$. If $\vec{A}$ is as in Proposition 1.11 (and $T = {}^{<\omega}\omega$), the function $n : \omega^\omega \to \mathbf{N}$ is total. We shall consider classes of sets for which n is "simple".

Definition 1.12. $B^*(\vec{A})$. Let T be a game tree, and let $\vec{A} = \langle A_\alpha | \alpha < \omega^2 \rangle$ be an ω^2 sequence of subsets of $[T]$.

If n is a total function from the reals into $\mathbf{N}$, let

$$n^*\left(\vec{A}\right) =_{df} \left\{x \in [T]\, | x \in dk\left(\vec{A}_{\omega \cdot n(x)}\right)\right\}.$$

For $B \subseteq [T] \times \omega$, let:

$$n_B(x) =_{df} \begin{cases} \mu n B(x,n) & \text{if } \exists n B(x,n) \\ 0 & \text{otherwise.} \end{cases}$$

$$B^*\left(\vec{A}\right) =_{df} n^*_B\left(\vec{A}\right). \qquad \blacksquare$$

$$\begin{aligned} x \in B^*\left(\vec{A}\right) &\Leftrightarrow x \in dk\left(\vec{A}_{\omega \cdot n_B(x)}\right) \\ &\Leftrightarrow \exists n \left[B(x,n) \wedge \forall m < n \neg B(x,m) \wedge \exists \alpha < \omega \cdot n \left(\alpha \text{ is odd } \wedge x \in \bigcap_{\beta<\alpha} A_\beta \setminus A_\alpha\right)\right] \end{aligned}$$

For $x \in B^*\left(\vec{A}\right)$, not only must $\mu\alpha\,(x \notin A_\alpha)$ be odd (and exist), α must be less than $\omega \cdot n_B(x)$ (that is, x must also "fall out" of $\vec{A}$ by stage $n_B(x)$ for player I). When $\vec{A}$ witnesses some set is $\omega^2 - \mathbf{\Pi}^1_1$ and $n_B(x)$ is the constant m, $B^*\left(\vec{A}\right)$ is the $\omega \cdot m - \mathbf{\Pi}^1_1$ set $dk\left(\vec{A}_{\omega \cdot m}\right)$.

We now define the lightface $(\Gamma)^*$ subsets of ${}^\omega\omega$ and its boldface analogue $(\Gamma)^{**}$.

Definition 1.13. $(\Gamma)^*$, $(\Gamma)^{**}$.

(i) For $\Gamma \subseteq \wp\left(({}^\omega\omega) \times \omega\right)$,

$$(\Gamma)^* =_{df} \left\{B^*\left(\vec{A}\right) | B \in \Gamma \text{ and } \vec{A} \text{ witnesses that some set of reals is } \omega^2 - \Pi^1_1\right\}.$$

(ii) Let T be a game tree.

$$(\Gamma)^{**} =_{df} \left\{B^*\left(\vec{A}\right) | B \in \Gamma,\ B \subseteq [T] \times \omega,\ \vec{A} \text{ is an } \omega^2 \text{ sequence of } \mathbf{\Pi}^1_1 \text{ subsets of } [T]\right\}.$$

B and $\vec{A}$ are said to *witness* that $B^*\left(\vec{A}\right) \in (\Gamma)^*$ [respectively $\in (\Gamma)^{**}$] when B and $\vec{A}$ are as in (i) [respectively, (ii)]. △

It is easy to show

$$\left(\Sigma^1_1\right)^* = \omega^2 - \Pi^1_1$$

and

$$\Delta(\omega^2 - \Pi^1_1) = \left(\{B \in \Sigma^1_1 | B \subseteq ({}^\omega\omega) \times \omega \text{ and } \forall x \exists n B(x,n)\}\right)^*.$$

These are shown in Proposition 1.10 of [Du95] and their boldface analogues hold by the same proof. Besides $(\Gamma)^*$ for various values of Γ (e.g. $\Gamma = \Pi^0_\alpha, \Sigma^0_\alpha, \Pi^1_1$), we are also interested in the slightly larger class $(\Gamma)^*_+$ which we now define.

Definition 1.14. $(\Gamma)^*_+$, $(\Gamma)^{**}_+$.

(i) Let T be a game tree, $B \subseteq [T] \times \omega$, $\vec{A}$ be an ω^2 sequence of subsets of $[T]$, and $D \subseteq [T]$. Let

$$B^*\left(\vec{A}, D\right) =_{df} B^*\left(\vec{A}\right) \cup \{x \in D | \forall n \neg B(x,n)\}.$$

If $\vec{D}$ is a sequence of subsets of $[T]$, let

$$B^*\left(\vec{A}, \vec{D}\right) =_{df} B^*\left(\vec{A}, dk\left(\vec{D}\right)\right).$$

(ii) For $\Gamma \subseteq \wp(({}^\omega\omega) \times \omega)$, $(\Gamma)^*_+$ is the collection of all $B^*\left(\vec{A}, D\right)$ for which $B \in \Gamma$, $\vec{A}$ witnesses some set of reals is $\omega^2 - \Pi^1_1$, and D is a $< \omega^2 - \Pi^1_1$ set of reals. In this case, [if $\vec{D}$ witnesses that $D \in < \omega^2 - \Pi^1_1$] we say that B, $\vec{A}$, and D [respectively $\vec{D}$] *witness* that $B^*\left(\vec{A}, D\right)$ [respectively $B^*\left(\vec{A}, \vec{D}\right)$] is $(\Gamma)^*_+$.

(iii) Let T be a game tree.

$$(\Gamma)^{**}_+ =_{df} \Big\{B^*\left(\vec{A}, D\right) | B \in \Gamma,\ B \subseteq [T] \times \omega,$$
$$\vec{A} \text{ is an } \omega^2 \text{ sequence of } \mathbf{\Pi}^1_1 \text{ subsets of } [T], \text{ and } D \subseteq [T] \text{ is } < \omega^2 - \mathbf{\Pi}^1_1\Big\}.$$

We define "witnessing a set is $(\Gamma)^{**}_+$" in the obvious manner. △

Clearly $(\Gamma)^* \subseteq (\Gamma)^*_+$, and for a fixed game tree, $(\Gamma)^{**} \subseteq (\Gamma)^{**}_+$. The classes $\left(\Pi^0_k\right)^*$, $\left(\Pi^0_k\right)^*_+$, $\left(\Pi^1_1\right)^*$, $\left(\Pi^1_1\right)^*_+$ all lie near the bottom of $\Delta\left(\omega^2 - \Pi^1_1\right)$. (See Propositions 1.11 and 1.12 of [Du95].) Between each $\left(\Pi^0_k\right)^*_+$ and $\left(\Pi^0_{k+1}\right)^*$ is a rich hierarchy of classes $(\Gamma)^*$ and $(\Gamma)^*_+$. Also such hierarchies exist above $\left(\Pi^1_1\right)^*_+$, again near the bottom of $\Delta\left(\omega^2 - \Pi^1_1\right)$. We next introduce the classes in these hierachies.

Definition 1.15. $(\vec{B})$. Let T be a game tree. Let $k \in ON$ and let $\vec{B} = \langle B_i | i < k\rangle$ be such that each $B_i \subseteq [T] \times \omega$.

$\left(\vec{B}\right)(x,n) \Leftrightarrow_{df}$ there exists i such that (i,n) is lexicographically least such that $B_i(x,n)$.

$$(\langle\Gamma_i | i < k\rangle) =_{df} \left\{ \left(\vec{B}\right) | \forall i < k\, B_i \in \Gamma_i \right\}. \qquad \triangle$$

The earlier B_i's are given priority in determining whether $\left(\vec{B}\right)(x,n)$ holds. In particular, n such that $\left(\vec{B}\right)(x,n)$ may not be least such that $\exists i B_i(x,n)$. Instead we look for the least i such that $\exists n B_i(x,n)$ and then for the least n such that $B_i(x,n)$.

Notation. If k is finite, we sometimes write $(B_0, B_1, B_2, \cdots, B_{k-1})$ for $\left(\vec{B}\right)$ and $(\Gamma_0, \Gamma_1, \Gamma_2, \cdots, \Gamma_{k-1})$ for $\left(\vec{\Gamma}\right)$. For $k_1, k_2, \cdots, k_n \in ON$,

$$(k_1 * \Gamma_1, k_2 * \Gamma_2, \ldots, k_n * \Gamma_n) =_{df} \left(\left\langle \tilde{\Gamma}_i | i < k_1 + k_2 + \cdots + k_n \right\rangle\right)$$

where

$$\tilde{\Gamma}_i = \begin{cases} \Gamma_1 & \text{if } i < k_1 \\ \Gamma_j & \text{if } k_1 + k_2 + \cdots + k_{j-1} \le i < k_1 + k_2 + \cdots + k_j. \end{cases}$$

In particular, for $k_1, k_2, \cdots, k_n$ finite,

$$(k_1 * \Gamma_1, k_2 * \Gamma_2, \cdots, k_n * \Gamma_n) \text{ is the class } (\Gamma_1, \Gamma_1, \cdots, \Gamma_1, \Gamma_2, \Gamma_2, \cdots, \Gamma_2, \cdots, \Gamma_n, \Gamma_n, \cdots, \Gamma_n),$$

where Γ_1 is repeated k_1 times, Γ_2 k_2 times, Γ_3 k_3 times, etc.

We also let

$$(\cdots, < \gamma * \Gamma, \cdots) =_{df} \bigcup_{\alpha < \gamma} (\cdots, \alpha * \Gamma, \cdots),$$

from which it immediately follows that:

$$(\cdots, < \gamma * \Gamma, \cdots)^* = \bigcup_{\alpha<\gamma} (\cdots, \alpha * \Gamma, \cdots)^* \text{ and } (\cdots, < \gamma * \Gamma, \cdots)^*_+ = \bigcup_{\alpha<\gamma} (\cdots, \alpha * \Gamma, \cdots)^*_+.$$

Remark 1.16. Note that if $\vec{B}$ is one-element sequence $\langle B\rangle$, then $\left(\vec{B}\right)$ is typically more complex than B: $(\langle B\rangle)(x,n) \Leftrightarrow B(x,n) \wedge \forall m < n \neg B(x,m)$. Suppose we had defined $\left(\vec{B}\right)$ as follows:

$$\left(\vec{B}\right)(x,n) \Leftrightarrow \exists i \left(\forall j < i \forall m \neg B_j(x,m) \wedge B_i(x,n)\right).$$

Then $\left(\vec{B}\right) = B$ when $\vec{B}$ is the one-element sequence $\langle B\rangle$. Also, both definitions of $\left(\vec{B}\right)$ result in the same classes $(\Gamma)^*$ and $(\Gamma)^*_+$.

We are sloppy concerning not deleting extra parenthesis in denoting $\left(\vec{\Gamma}\right)$. For instance, if $\Gamma = (\langle\Gamma_i | i < \beta\rangle)$, we freely write (Γ, Γ_β), $(\Gamma, \Gamma_\beta)^*$, and $(\Gamma, \Gamma_\beta)^*_+$ for $(\langle\Gamma_i | i \le \beta\rangle)$, $(\langle\Gamma_i | i \le \beta\rangle)^*$, and $(\langle\Gamma_i | i \le \beta\rangle)^*_+$.

We are interested in the determinacy strength of the classes $\left(\vec{\Gamma}\right)^*$ and $\left(\vec{\Gamma}\right)^*_+$ for which the individual $\Gamma_i \in \left\{\Pi^0_\alpha, \Sigma^0_\alpha, \Pi^1_1 | \alpha < \omega_1^{CK}\right\}$. However, we shall need to consider the codes for the boldface analogues of these classes (on game trees other than ${}^{<\omega}\omega$). We next define such codes.

Definition 1.17. Let T be a game tree. If $\vec{B} = \langle B_\alpha | \alpha < \gamma\rangle$ is a $\mathbf{\Pi^1_1}$ sequence (of subsets of $[T] \times \omega$) with code (x, T, S_1), $\vec{A} = \left\langle A_\alpha | \alpha < \omega^2 \right\rangle$ is a $\mathbf{\Pi^1_1}$ sequence (of subsets of $[T]$) with code (T, S_2), and $\vec{D} =$

$\langle D_\alpha | \alpha < \beta \rangle$ is a $\mathbf{\Pi^1_1}$ sequence (of subsets of $[T]$) with code (T, S_3) and $\beta < \omega^2$, then (x, T, S_1, S_2) [respectively (x, T, S_1, S_2, S_3)] is a *code* for $\left(\vec{B}\right)^* \left(\vec{A}\right)$ [respectively $\left(\vec{B}\right)^* \left(\vec{A}, \vec{D}\right) = \left(\vec{B}\right)^* \left(\vec{A}, dk \left(\vec{D}\right)\right)$].

If $\omega_1^{CK}(x) = \omega_1^{CK}$, we drop the x in the above codes. △

We have not concerned ourselves with defining, for instance, codes for $\left(\mathbf{\Sigma}^1_1\right)^{**}$ sets, even though an appropriate definition is clear.