

DETERMINACY AND
MULTIPLAYER
GAMES

by

Christine Lee McKenna

Bachelor of Arts
University of Nevada, Las Vegas
1996

A thesis submitted in partial fulfillment
of the requirements for the

Master of Science Degree
Department of Mathematical Sciences
College of Sciences

Graduate College
University of Nevada, Las Vegas
December 2005

TABLE OF CONTENTS

ABSTRACT	iii
ACKNOWLEDGMENTS	v
PRELIMINARIES AND NOTATION	vi
CHAPTER 1 NONDETERMINACY OF MULTIPLAYER GAMES	1
CHAPTER 2 DETERMINACY OF THREE-PLAYER BIASED GAMES:	
A WARM-UP	17
CHAPTER 3 DETERMINACY OF THREE-PLAYER BIASED GAMES	40
CHAPTER 4 SHOWING THE DETERMINACY OF BIASED GAME (CHAPTER 3)	
IS OPTIMAL.....	68
CHAPTER 5 PERMUTING THE PLAYER'S ROLES	86
CHAPTER 6 DETERMINACY OF MULTIPLAYER BIASED GAMES	104
BIBLIOGRAPHY	130
VITA.....	131

ABSTRACT

Determinacy and Multiplayer Games

by

Christine Lee McKenna

Dr. Derrick DuBose, Examination Committee Chair
Associate Professor of Mathematics
University of Nevada, Las Vegas

In the field of set theory, two-player infinite games of perfect information are well studied. The determinacy of various classes of such games have led to many important results. Furthermore, such determinacy follows from large cardinal axioms. In this thesis, we are instead interested in such infinite games with more than two players. With the study of two-player games being so fruitful, why aren't such infinite games studied with more than two players?

One difficulty in proving determinacy is that players need not play in any reasonable manner: A player may actually play a move that immediately results in a winning strategy or even an instant win for another player, even when such a move need not be played. We note that this leads to nondetermined games of extremely low complexity with three players, four players, five players, etc. However, we obtain determinacy of multiplayer games in which all but one player has an open payoff set and in which certain conditions are placed on certain player's moves: certain players will not be allowed to make a move that immediately results in a winning strategy for certain other players whenever such a move exists.

Table of Contents is on another page
The table of contents is page four

ACKNOWLEDGMENTS

I would like to thank Dr. Derrick DuBose for his time, help, and patience. I would also like to thank Dr. Douglas Burke, Dr. Peter Shiue, and Dr. Jay Coughtry for their time and helpful input. Lastly, I would like to thank my husband, Michael, and my daughters, Judith and Rebekah, for their support.

ABSTRACT

Determinacy and Multiplayer Games

by

Christine Lee McKenna

Dr. Derrick DuBose, Examination Committee Chair
Associate Professor of Mathematics
University of Nevada, Las Vegas

A natural requirement in any game is to require that the players cannot play any move that results immediately in a winning strategy for another player. Of course, one needs that there is always a move to play consistent with this requirement in order for play to continue. We show that one of the players has a winning strategy in any infinite finite-player game of perfect information such that: (1) the game satisfies the above “natural” requirement, (2) a move is always available to play that does not violate the requirement, and (3) all except possibly one of the players has an open payoff set (i.e. at most one player has a payoff set that is not open).

We actually show a stronger result that implies this. In the thesis, we weaken the “natural” requirement above so that only certain players can’t play a move that results immediately in a particular type of winning strategy for certain other players. It is well known, at least in set theory, that for a player X with open payoff: (i) ordinals of position for player X can be defined, and used to define a w.s. for player X when a position is reached that has such an ordinal, and (ii) that players other than X can use ordinals of positions of X to avoid reaching a position at which X has a winning strategy when a

position does not have such an ordinal. We use a restricted version of ordinals of positions that still satisfies (i) and (ii). Our “natural” requirement is weakened so that only certain players are restricted from making moves that result in a position that is in the ordinals of certain other players. The final result is striking in that we also provide three-player, four-player, \aleph_1 games of perfect information with extremely low complexity in which no player has a winning strategy.

ABSTRACT

Determinacy and Multiplayer Games

By

Christine Lee McKenna

Dr. Derrick DuBose, Examination Committee Chair
Associate Professor of Mathematics
University of Nevada, Las Vegas

In the field of set theory, two-player infinite games of perfect information are well studied. The determinacy of various classes of such games have led to many important results (Lebesgue measurability, the Baire property, and selection principles for certain sets). Furthermore, such determinacy follows from large cardinal axioms. In this thesis, we are instead interested in such infinite games with more than two players. With the study of two-player games being so fruitful, why aren't such infinite games studied with more than two players?

One difficulty in proving determinacy is that players need not play in any reasonable manner: A player may actually play a move that immediately results in a winning strategy or even an instant win the another player, even when such a move need not be played. We note (in Chapter 1) that this leads to nondetermined games of extremely low complexity with three players, four players, five players, etc. However, we obtain determinacy of multiplayer games in which all but one player has an open payoff set and in which certain conditions are placed on certain player's moves: certain players will not

be allowed to make a move that immediately results in a winning strategy for certain other players whenever such a move exists.

1st rough draft, very dry Do Not Use

ABSTRACT

Determinacy and Multiplayer Games

By

Christine Lee McKenna

Dr. Derrick DuBose, Examination Committee Chair
Associate Professor of Mathematics
University of Nevada, Las Vegas

We consider certain three-player infinite games of perfect information which satisfy non-helping conditions. We investigate those open games in which two players are assigned open payoff sets. We use ordinals of a position to define the moves of the players. We define non-helping conditions which restrict the moves of players from helping their opponents, especially those players with open payoff sets. We show that these three-player biased open games which satisfy non-helping conditions are determined. We examine the question as to whether our non-helping conditions are optimal. We investigate other players having open payoff sets, that the order of assigning open payoff sets does not matter, as long as the non-helping conditions also reflect this change. We then generalize our result of three-player open games are determined to an $(n + 1)$ -player open biased game is determined.

PRELIMINARIES AND NOTATION

Our research questions arise from the field of determinacy and set theory in which typically one studies two-person infinite games of perfect information. The determinacy of various classes of such games have led to many important results (Lebesgue measurability, the Baire property, and selection principles for certain sets). Furthermore, such determinacy follows from large cardinal axioms and level-by-level correspondences between large cardinals and determinacy of various definable classes of games are well-known.

First let's review infinite games of perfect information. We shall describe a play in such a game, the length of the game, the payoff sets for the game, etc. An infinite game of perfect information on a set X is *infinite* since there are infinitely many moves, which can be denoted by $f(0), f(1), f(2), f(3), \dots$ that are played. These moves are to be chosen from the set X , i.e. $f(i) \in X$ for every $i \in \omega$.¹ If a game has a finite number n of players, then the first player plays $f(0), f(n), f(2n), f(3n), \dots$, and more generally, the k^{th} player plays $f(k-1), f(n+k-1), f(2n+k-1), f(3n+k-1), \dots$. A play of an infinite game of perfect information with length ω can be thought of as a function f from ω into X , that is, $f \in X^\omega$. The function f is defined by the players "taking turns" choosing an element from the set X (to play as moves). In this thesis, we are interested in games in which the moves are from the natural numbers. Player I chooses $f(0) \in X$

¹ It is standard that ω denotes the set of natural numbers: $\{0, 1, 2, 3, \dots\}$.

first, then player II chooses $f(1) \in X$, player III chooses $f(2) \in X$, and this continues until the last player (the n^{th} player) selects $f(n-1) \in X$. Together these first n moves make up the first inning of play. Following the first inning of play, the second inning of play starts with player I selecting $f(n) \in X$, and continues in the obvious manner until the last player plays $f(n-1) \in X$. The i^{th} inning of play is defined analogously.

Associated with this game are *payoff sets* for the players. Each payoff set is a collection of functions from ω into X . Suppose A_1, A_2, \dots, A_n respectively denote the payoff sets of player I, II, III, etc. Then each $A_i \subseteq X^\omega$. Player I wins the game if the resulting play f is in I's payoff set A_1 ; that is, player I wins the game iff $f \in A_1$. More generally, the k^{th} player wins the game iff $f \in A_k$. We do not allow for ties so that we require the payoff sets to be pairwise disjoint, i.e. $i \neq j \Rightarrow A_i \cap A_j = \emptyset$ and $\bigcup_{i=1}^n A_i = X^\omega$.

The game is said to be of *perfect information* since at any point in the game, each player has full knowledge of all the previous moves in the game and of each player's payoff set. This fact is important when describing strategies and winning strategies for a given player. σ is a strategy for player I if σ defines a move for player I in terms of the previous moves in the game. More generally, σ is a strategy for the k^{th} player if σ defines a move for the k^{th} player in terms of the previous moves in the game. "A play $f = (f(0), f(1), K, f(n), K, f(2n), K)$ is according to a strategy σ for the k^{th} player" has the obvious meaning:

$$f(j) = \sigma(f(0), f(1), K, f(j-1)) \text{ whenever } f(j) \text{ is a move for the } k^{\text{th}} \text{ player.}$$

In that case, $j = in + k - 1$, where $i+1$ is the inning in which $f(j)$ is played. Hence:

$$f(in+k-1) = \sigma(f(0), f(1), \dots, f(in+k-2)) \text{ for } i \in \omega.$$

A strategy σ is a *winning strategy* for the k^{th} player if for all $f \in X^\omega$ that is according to σ , $f \in A_k$ (in which case, the k^{th} player wins the game). A game is said to be *determined* if one of the players has a winning strategy for the game.

We are naturally interested in the complexity of the payoff sets, as determinacy is more likely to hold for games with payoff sets of low complexity. In two-player infinite games of perfect information, determinacy has been proven for games with somewhat “complex” payoff sets.

One difficulty in proving determinacy is that players need not play in any reasonable manner: A player may actually play a move that immediately results in a winning strategy or even an instant win for another player, even when such a move need not be played. This leads to nondetermined games of extremely low complexity with three players, four players, five players, etc. (see Chapter 1). However, in this thesis, we still obtain determinacy of certain multiplayer games by introducing some restrictions on certain players’ moves. Certain players will not be allowed to make a move that immediately results in a winning strategy for certain other players. For the multiplayer games for which we obtain determinacy in this thesis, the payoff sets of all but one player will be open (defined below).

A payoff set $A \subseteq X^\omega$ is *open* iff there is a collection $\{\overset{r}{p}_i \mid i \in I\}$ of positions in the game such that A is exactly all plays that extend some position in D :

$$f \in A \text{ iff } \exists \overset{r}{p} \in D (\overset{r}{p} \subseteq f).$$

A game is *open* for a player if that player has an open payoff set. In this case, if such a player wins the game, this is known at some position in the game, as that player had

extended some position in D . Further, the converse holds, as a player with open payoff must extend some position in D or otherwise lose the game. Typically payoff sets are more complex than open, and one doesn't know if the play is in a payoff set until the entire play is complete.

Gale and Stewart [GS53] showed that all infinite two-player games of perfect information which are open for one of the players are determined. Many other interesting results are well-known about the determinacy of two-player games. Excellent references for material on determinacy include: Chapter 6 of *Descriptive Set Theory* by Yiannis N. Moschovakis [Mo80], Chapter 6 of *The Higher Infinite*, Second Edition, by Akihiro Kanamori [Ka03], *Classical Descriptive Set Theory* by Alexander S. Kechris [Ke95], Donald A. Martin's upcoming book on determinacy [Ma ∞], and the paper "Long Games" by John R. Steel [St88].

It is well-known that associated with an open payoff set for a player X , one can define ordinals of positions, ORD_X . This is presented in Chapter 2. ORD_X can be used to define a winning strategy for player X whenever a position is reached that is in ORD_X . For certain players X and Y , we shall require that player Y plays moves that are not in ORD_X whenever such a move exists. (This may be viewed as a natural requirement, because otherwise if player Y plays a move resulting in a position with an ORD_X -value, then player X has a winning strategy for the remainder of the game.) We obtain determinacy of multiplayer games in which all but one player has open payoff, by adding this type of

“non-helping” requirements to the game.² A corollary to the result is:

In Chapter 3, we prove determinacy of such three-player games which are open for players I and II by adding three such “non-helping” conditions. In Chapter 4 we show that determinacy cannot be proved if we drop any one of these three conditions. In Chapter 5, we show that the Chapter 3 result carries over when the three-player game is open for any two of the players. We generalize the Chapter 5 determinacy result to any finite number of players in Chapter 6. In Chapter 1, we present nondetermined multiplayer games of low complexity. In Chapter 2, we present the definition of ordinals of positions and use it to prove some determinacy as a warm-up for the later results.

The following is a nice corollary to the determinacy results in this thesis:

Corollary 6.6. Determined is any infinite finite-player game of perfect information in which:

- 1 At most one player has a payoff set that is not open,
- 2 At every position, there is a move m such that at the resulting position, not player other than possibly the player making the move m has a winning strategy, and
- 3 Each player is required to make such a move m .

We actually show a stronger determinacy result (see Theorem 6.3) than Corollary 6.6. But the above corollary is more easily cited, as it doesn’t require the reader to look at the definitions of our “non-helping” conditions!³

² We emphasize “type” because we shall need to use a “restricted” version of ORD_x , instead of ORD_x in many of our non-helping requirements. Our restricted version of ORD_x is obtained by restricting the quantifiers in the usual definition of ORD_x to a certain set (such a set will be the collection of positions whose initial segments are all out of the “restricted” ordinal for another player).

³ The non-helping conditions though “give” stronger results.

CHAPTER 1

NONDETERMINACY OF MULTIPLAYER GAMES

Even though the axiom of choice implies that some two player games are not determined, it is well known that two-player games of perfect information in which the payoff sets are sufficiently definable are determined. In fact, Donald A. Martin [Ma75] proved all Borel games are determined, and he and others proved that the existence of very large cardinals implies the determinacy of projective sets [MS89]. In fact to obtain a nondetermined two-player game requires the use of the Axiom of Choice. One might anticipate a similar situation regarding determinacy for three-player games. This turns out not to be the case! In this section we prove that some very simple three player games are not determined. Initially we present a nondetermined three-player game with only one move, and then present a nondetermined three-player game with at most one inning of play. We then generalize these results to games with more players.

Theorem 1.1. There exists a nondetermined three-player game. In fact, \forall set E with $|E| \geq 2$ there exists a nondetermined three-player game on E with exactly one move and in which player I has empty payoff.

Proof. Pick an arbitrary set E with $|E| \geq 2$. We describe a three player game G on E and show G is not determined. Player I has empty payoff. Let $a \in E$. Player II wins G iff player I's first move is a . Therefore player III wins G iff player I's first move is not a .

Claim. This game G is not determined.

Since player I has empty payoff, any play $\overset{1}{y}$ according to any strategy for player I cannot be a win for player I. Therefore, player I has no winning strategy.

Any strategy σ , for player II cannot be a winning strategy since any play $\overset{1}{y}$ that is according to σ in which player I's first move is not a is a loss for player II. Therefore player II has no winning strategy.

Similarly, player III has no winning strategy since any play $\overset{1}{y}$ in which player I's first move is a is a loss for player III, regardless of whether $\overset{1}{y}$ is according to some strategy for player III.

Thus the game is not determined. , (Theorem 1.1)

In Theorem 1.1, player I could not win, but decided the winner through his move. Next we show that any one of the players could be the decision maker of the game in the first inning of play.

Theorem 1.2. Let E be a set with a least two elements. $\forall X \in \{I, II, III\}$ there exists a nondetermined three-player game on E with at most one inning of play and in which player X has empty payoff.

Proof. Pick an arbitrary set E with $|E| \geq 2$. We describe a three player game G on E and show G is not determined. Let $a \in E$. Let X, Y and Z be players I, II and III (in any order). Player X has empty payoff set. Player Y wins G iff player X 's first move is a . Player Z wins G iff player X 's first move is not a .

Claim. This game G is not determined.

Since player X has empty payoff, any play $\overset{1}{y}$ according to any strategy for player X cannot be a win for player X . Therefore, player X has no winning strategy.

Any strategy σ , is not a winning strategy for player Y since any play $\overset{1}{y}$ that is according to σ in which player X 's first move is not a , is a loss for player Y . Therefore, player Y has no winning strategy.

Similarly, player Z has no winning strategy since any play $\overset{1}{y}$ in which player X 's first move is a is a loss for player Z , regardless of whether $\overset{1}{y}$ is according to some strategy for player Z .

Thus the game is not determined. , (Theorem 1.2)

The proof of Theorem 1.2 goes through even if we add additional players who have empty payoff. If E is a set with at least two elements and κ is any ordinal greater than or equal to three, then there exists a nondetermined κ -player game on E in which only two players have nonempty payoff sets and at least one player with empty payoff makes a move. The player with the move decides the winner between the two players with nonempty payoff. We now state this result in Theorem 1.3 and shall use this result in Theorem 1.5.

Theorem 1.3. Let E be any set with at least two elements. Let κ be any ordinal ≥ 3 . Then there exists a nondetermined κ -player game on E (in which exactly two players have nonempty payoff and in which one player with empty payoff makes a move).

Proof. Pick an arbitrary set E with $|E| \geq 2$. Let κ be any ordinal ≥ 3 . We describe a κ -player game G on E and show G is not determined. Let $a \in E$. Pick any three players X, Y , and Z from the κ players. Let X have empty payoff and let player X make a move m . Player Y wins iff $m = a$. Player Z wins iff $m \neq a$. Since only player Y and Z have nonempty payoff, no other player can win. Since player X decides which of the players Y or Z wins, neither player Y nor Z has a winning strategy (as in proof of Theorem 1.2).
, (Theorem 1.3)

In the games from Theorems 1.1, 1.2 and 1.3, only two players had nonempty payoff. Also in those games we had a player who could not win (due to having an empty payoff set) decide the winner through his move, making all other moves irrelevant. One can use this to get, for any ordinal κ greater than or equal to three, nondetermined κ -player games in which all but one player has nonempty payoff.

Theorem 1.4. Let κ be any ordinal ≥ 3 . There exists a nondetermined κ -player game. In fact, for any set E with $|E| \geq \kappa$ there exists a nondetermined κ -player game on E with exactly one move.

Proof. Let κ be any ordinal ≥ 3 and let $\{e_i \mid i < \kappa\} \subseteq E$. Let $\langle X_i \mid i < \kappa \rangle$ be the sequence of κ players, in order of play, so that the player X_0 is the first player. Let player X_0 have empty payoff. If his first move is e_i for $2 \leq i < \kappa$ then player X_i wins. Otherwise, player X_1 wins. Player X_0 can make any player other than himself win through his first move. Therefore, since player X_0 can't win, no player can have a winning strategy (by the argument in Theorem 1.1). , (Theorem 1.4)

A slight adjustment to the proof of Theorem 1.4 shows that for any set E , there exists a nondetermined $(|E|+1)$ -player game on E : Again let player X_0 have empty payoff; but for $1 \leq i < \kappa$, let player X_i win iff player X_0 plays e_i . Otherwise player X_κ wins.

In Theorem 1.5, we provide nondetermined κ -player games in which all but two payoff sets are empty. In the nondetermined games of Theorem 1.4, we eliminated having so many empty payoff sets (only one empty payoff set) but in exchange played on a large set (if the number of players is large). We next consider the possibility of playing on a small set (possibly of size 2) and having “several” nonempty payoff sets (possibly $2^{\# \text{ of players making a move}}$).

Next we present a generalized proof of nondetermined κ -player games for any ordinal κ greater than or equal to three in which we have at least three players making relevant moves. The payoff sets are defined by the permutations of the moves made by three named players from a set of at least two elements. These games have a possibility of more than one player being able to determine who wins, and that player need not have an empty payoff.

Theorem 1.5. Let E be any set with $|E| \geq 2$ and let κ be any ordinal ≥ 3 . There is a nondetermined κ -player game on E in which at least three players each make (at least) one move and at least two and not more than 2^3 players have nonempty payoff sets.¹

Proof. Pick an arbitrary set E with $|E| \geq 2$. We shall describe a game G that depends on E with κ players in which the number of players with nonempty payoff sets is at least two but no more than eight. At least three players will each make (at least) one move. All moves are from the set E . The situation in which two players have nonempty payoff is handled as a special (separate) case.

If exactly two players have nonempty payoff, then the proof of Theorem 1.3 gives us our result. In this case, one of the three players making a move has empty payoff and that player decides through his move which of the two players with nonempty payoff wins. By the proof of Theorem 1.3, this game is not determined. Since the case in which exactly two players have nonempty payoff is done, we shall assume we have at least three players with nonempty payoff for the remainder of the proof. The nonempty payoff sets for this case will be defined after we provide suitable information.

Let $\langle X_i \mid i < \kappa \rangle$ be the sequence of κ players in order of play. Since there are at least three players to make a move, $\exists \alpha, \beta, \gamma$ such that $0 \leq \alpha < \beta < \gamma < \kappa$, and players X_α , X_β , and X_γ each make at least one move. Let $f(w_\alpha)$, $f(y_\beta)$, and $f(z_\gamma)$ respectively

¹ If exactly two players have nonempty payoff sets, this theorem can be strengthened (see Theorem 1.3) in that we only need a third player to make a move (not three players total to make moves).

denote a move of players X_α , X_β , and X_γ .²

Let S designate the number of players with nonempty payoff sets. Recall $3 \leq S \leq 8$ (and that we have already provided the proof for $S = 2$). Let $X_{p_1}, X_{p_2}, X_{p_3}, \dots, X_{p_S}$ enumerate those players having nonempty payoff sets, in any order for which $p_1 = \alpha$ if X_α has nonempty payoff (any order if X_α has empty payoff). The nonempty payoff sets for these players will be defined in terms of sets A_r , which we define next. Recall that $|E| \geq 2$ so that $\exists a, b \in E$ such that $a \neq b$.

Let

$$\begin{aligned} A_1 &=_{\text{df}} \{ \text{plays } f \mid f(w_\alpha) = a \wedge f(y_\beta) = a \wedge f(z_\gamma) = a \} \\ A_2 &=_{\text{df}} \{ \text{plays } f \mid f(w_\alpha) = a \wedge f(y_\beta) = a \wedge f(z_\gamma) \neq a \} \\ A_3 &=_{\text{df}} \{ \text{plays } f \mid f(w_\alpha) = a \wedge f(y_\beta) \neq a \wedge f(z_\gamma) = a \} \\ A_4 &=_{\text{df}} \{ \text{plays } f \mid f(w_\alpha) = a \wedge f(y_\beta) \neq a \wedge f(z_\gamma) \neq a \} \\ A_5 &=_{\text{df}} \{ \text{plays } f \mid f(w_\alpha) \neq a \wedge f(y_\beta) = a \wedge f(z_\gamma) = a \} \\ A_6 &=_{\text{df}} \{ \text{plays } f \mid f(w_\alpha) \neq a \wedge f(y_\beta) = a \wedge f(z_\gamma) \neq a \} \\ A_7 &=_{\text{df}} \{ \text{plays } f \mid f(w_\alpha) \neq a \wedge f(y_\beta) \neq a \wedge f(z_\gamma) = a \} \\ A_8 &=_{\text{df}} \{ \text{plays } f \mid f(w_\alpha) \neq a \wedge f(y_\beta) \neq a \wedge f(z_\gamma) \neq a \} \end{aligned}$$

In assigning nonempty payoff sets to the S players in terms of the A_r 's, consider two cases: $S \geq 5$ and $S \leq 4$:

Case 1: If $S \geq 5$, then:

- (i) Payoff set $(X_{p_m}) = A_m$ for $m < S$, and

²Where $w_\alpha = q_\alpha \kappa + \alpha$ for some $q_\alpha \in \{0, 1, 2, K\}$, where $y_\beta = q_\beta \kappa + \beta$ for some $q_\beta \in \{0, 1, 2, K\}$, and where $z_\gamma = q_\gamma \kappa + \gamma$ for some $q_\gamma \in \{0, 1, 2, K\}$.

(ii) Payoff set $(X_{P_S}) = A_S \cup A_{S+1} \cup L \cup A_8$.

Case 2: If $S \leq 4$, then:

(i) Payoff set $(X_{P_m}) = A_m$ for $m < S - 1$, and

(ii) Payoff set $(X_{P_{S-1}}) = A_{S-1} \cup A_S \cup L \cup A_4$, and

(iii) Payoff set $(X_{P_S}) = A_5 \cup A_6 \cup L \cup A_8$

In either case, payoff set (X_{P_1}) will be either a subset of

$$A_1 \cup A_2 \cup A_3 \cup A_4 = \{\text{plays } f \mid \text{player } X_\alpha \text{ plays } f(w_\alpha) = a\}$$

or a subset of

$$A_5 \cup A_6 \cup A_7 \cup A_8 = \{\text{plays } f \mid \text{player } X_\alpha \text{ plays } f(w_\alpha) \neq a\}^3$$

Therefore, if $p_1 \neq \alpha$, player X_α can make player X_{P_1} lose by playing an appropriate

value for $f(w_\alpha)$. If player X_α has nonempty payoff, then $p_1 = \alpha$ (by our definition of

p_1),

³ **Claim 1.** $\forall \alpha < \kappa$:

Payoff set $(X_\alpha) \subseteq A_1 \cup A_2 \cup A_3 \cup A_4$, or Payoff set $(X_\alpha) \subseteq A_5 \cup A_6 \cup A_7 \cup A_8$.

Proof. Pick an $\alpha < \kappa$.

Claim 1 clearly holds if payoff set $(X_\alpha) = \emptyset$. Also Claim 1 holds for Case 1-(i) and Case 2-(i), since payoff set $(X_{P_m}) = A_m$.

In Case 1-(ii), $S \geq 5$ and payoff set $(X_{P_S}) =_{\text{df}} A_S \cup L \cup A_8 \subseteq A_5 \cup L \cup A_8$.

In Case 2-(ii), $S \leq 4$ and payoff set $(X_{P_{S-1}}) =_{\text{df}} A_{S-1} \cup L \cup A_4 \subseteq A_1 \cup L \cup A_4$.

In Case 2-(iii), $S \leq 4$ and payoff set $(X_{P_S}) =_{\text{df}} A_5 \cup L \cup A_8$. , (Claim 1)

Since $A_1 \cup A_2 \cup A_3 \cup A_4 \subseteq \{\text{plays } f \mid f(w_\alpha) = a\}$ and $A_5 \cup A_6 \cup A_7 \cup A_8 \subseteq \{\text{plays } f \mid f(w_\alpha) \neq a\}$, by Claim 1 we get:

Claim 2. $\forall \alpha < \kappa$:

Payoff set $(X_\alpha) \subseteq \{\text{plays } f \mid f(w_\alpha) = a\}$ or Payoff set $(X_\alpha) \subseteq \{\text{plays } f \mid f(w_\alpha) \neq a\}$.

payoff set $(X_\alpha) = \text{payoff set } (X_{p_1}) = A_1 \subseteq \{\text{plays } f \mid f(y_\beta) = f(z_\gamma) = a\}$,

and players X_β and X_γ can make player X_α lose by one or both playing anything different from a . Players other than X_{p_1} for $1 \leq i \leq S$ have empty payoff and therefore always lose. Consequently, no player can have a winning strategy. For readers who require more details, we prove the following:

Claim. The game G is not determined.

Pick an arbitrary player X_i where $0 \leq i < \kappa$ and pick an arbitrary strategy s_i for player X_i . We will show s_i is not a winning strategy, i.e. there exists a play $\overset{I}{y}$ according to s_i such that $\overset{I}{y} \notin \text{payoff set } (X_i)$.

Case 1: Let $i \neq p_1$ for all $1 \leq i \leq S$, i.e. payoff set $(X_i) = \emptyset$.

Let $\overset{I}{y}$ be a play according to s_i . Since X_i has empty payoff, $\overset{I}{y}$ is a loss for player X_i . Therefore, s_i is not a winning strategy.

Case 2: Let $i = p_1$ for some $1 \leq i \leq S$ and $i \neq \alpha$. (Recall $S \geq 3$.)

Recall $X_{p_1}, X_{p_2}, X_{p_3}, \dots, X_{p_S}$ are the players with nonempty payoff. Since $i \neq \alpha$, player X_{p_1} is different from player X_α . By the definition of the payoff sets,

$$\text{payoff } (X_{p_1}) \subseteq \{\text{plays } f \mid f(w_\alpha) = a\} \text{ or } \text{payoff } (X_{p_1}) \subseteq \{\text{plays } f \mid f(w_\alpha) \neq a\}.$$

First, suppose payoff set $(X_{p_1}) \subseteq \{\text{plays } f \mid f(w_\alpha) = a\}$. Let $\overset{I}{y}$ be a play according to s_{p_1} in which player X_α plays $f(w_\alpha) \neq a$. Then $\overset{I}{y} \notin \text{payoff set } (X_{p_1})$ and therefore $\overset{I}{y}$ is a loss for player X_{p_1} . Therefore, s_{p_1} is not a winning strategy for player X_{p_1} .

Next, suppose payoff set $(X_{p_1}) \subseteq \{\text{plays } f \mid f(w_\alpha) \neq a\}$. Let $\frac{1}{y}$ be a play according to s_{p_1} in which player X_α plays $f(w_\alpha) = a$. Then $\frac{1}{y} \notin \text{payoff set } (X_{p_1})$ and therefore $\frac{1}{y}$ is a loss for player X_{p_1} . Therefore, s_{p_1} is not a winning strategy for player X_{p_1} .

Consequently, in both cases, since $i = p_1$, s_i is not a winning strategy.

Case 3: Let $i = p_1$, for some $1 \leq 1 \leq S$ and $i = \alpha$. (Recall $S \geq 3$.)

Since $\alpha = i = p_1$ for some $1 \leq 1 \leq S$, $1 = 1$ and $\alpha = i = p_1$ by definition of the p_1 .

Therefore player X_α (i.e. X_{p_1}) has payoff set A_1 .

Let $\frac{1}{y}$ be a play according to s_α in which player X_β plays $f(y_\beta) \neq a$ and/or player X_γ plays $f(z_\gamma) \neq a$. Since $A_1 \subseteq \{\text{plays } f \mid f(y_\beta) = f(z_\gamma) = a\}$, $\frac{1}{y} \notin \text{payoff set } (X_\alpha)$ and $\frac{1}{y}$ is a loss for player X_α . Therefore, s_α is not a winning strategy for player X_α (i.e. s_i is not a winning strategy).

In summary, s_i is not a winning strategy. Since s_i is an arbitrary strategy for player X_i and i was arbitrary, the game is not determined. , (Theorem 1.5)

In Theorem 1.5 we had at least three players each making a move in a nondetermined κ -player game played on a small set (possibly of size 2). In Theorem 1.6 we generalize a nondetermined κ -player game in which there are at least n players each making a move on a small set, (where n is greater than or equal to three), and in which there are at most 2^n nonempty payoff sets.

Theorem 1.6. Let n be a finite number ≥ 3 , let κ be an ordinal $\geq n$, and let E be any set with $|E| \geq 2$. There is a nondetermined κ -player game on E in which at least n players each make (at least) one move and at least two and not more than 2^n players have nonempty payoff sets.

Proof. The proof of this theorem is the same as that of Theorem 1.5: just replace 3 by n . We still handle the case in which exactly two players have nonempty payoff as a separate special case, as we did in Theorem 1.5.

Pick an arbitrary set E with $|E| \geq 2$. We shall describe a game G that depends on E with κ players in which the number of players with nonempty payoff sets is at least two but no more than 2^n . At least n players will each make (at least) one move. All moves are from the set E .

As indicated in the proof of Theorem 1.5, if exactly two players have nonempty payoff, then the proof of Theorem 1.3 gives us our result. In this case, since $n \geq 3$, one of the n players making a move has empty payoff and that player decides through his move which of the two players with nonempty payoff wins. By the proof of Theorem 1.3, this game is not determined. Since the case in which exactly two players have nonempty payoff is done, we shall assume we have at least three players with nonempty payoff for the remainder of the proof. The nonempty payoff sets for this case will be defined after we provide suitable information.

Let $\langle X_i \mid i < \kappa \rangle$ be the sequence of κ players in order of play. Since there are at least n players to make a move, $\exists j(1), j(2), \dots, j(n)$ such that

$0 < j(1) < j(2) < L < j(n-1) < j(n) < \kappa$, and for $1 \leq h \leq n$, player $(X_{j(h)})$ makes at least one move $f(w_{j(h)})$.⁴

Let S designate the number of players with nonempty payoff sets. Recall $3 \leq S \leq 2^n$ (as we have already provided the proof for $S = 2$). Let $X_{p_1}, X_{p_2}, X_{p_3}, L, X_{p_S}$ enumerate those players having nonempty payoff sets, in any order for which $p_1 = j(1)$ if $X_{j(1)}$ has nonempty payoff (any order if $X_{j(1)}$ has empty payoff). The nonempty payoff sets will be defined in terms of sets A_r , which we define next. Recall that $|E| \geq 2$ so that $\exists a, b \in E$ such that $a \neq b$. Therefore for $1 \leq h \leq n$ the move $f(w_{j(h)})$ of a player $X_{j(h)}$ will be either $f(w_{j(h)}) = a$ or $f(w_{j(h)}) \neq a$. $\forall \mathbf{R} = (R_1, R_2, R_3, L, R_n) \in \{=, \neq\}^n$ let

$$\begin{aligned} D_{\mathbf{R}} &= \{ \text{plays } f \mid f(w_{j(1)})R_1a, f(w_{j(2)})R_2a, f(w_{j(3)})R_3a, L, f(w_{j(n)})R_na \} \\ &= \{ \text{plays } f \mid \forall h \leq n f(w_{j(h)})R_ha \}. \end{aligned}$$

The set $\{ (=, R_2, R_3, R_4, L, R_n) \mid \text{each } R_h (2 \leq h \leq n) \text{ is either } = \text{ or } \neq \}$ has size 2^{n-1} so that 2^{n-1} is the number of $D_{(R_1, R_2, R_3, L, R_n)}$ in which R_1 is $=$, so let

$A_1, A_2, A_3, L, A_{2^{n-1}}$ enumerate these $D_{(=, R_2, R_3, R_4, L, R_n)}$'s. Similarly, the set

$\{ (\neq, R_2, R_3, R_4, L, R_n) \mid \text{each } R_h (2 \leq h \leq n) \text{ is either } = \text{ or } \neq \}$ has size 2^{n-1} , so let

$A_{2^{n-1}+1}, A_{2^{n-1}+2}, A_{2^{n-1}+3}, L, A_{2^n}$ enumerate these $D_{(\neq, R_2, R_3, R_4, L, R_n)}$'s.

In assigning nonempty payoff sets to the S players in terms of the A_r 's, consider two cases: $S \geq 2^{n-1} + 1$ and $S \leq 2^{n-1}$:

⁴ Where $w_{j(h)} = q_{j(h)}n + j(h)$ for some $q_{j(h)} \in \{0, 1, 2, K\}$.

Case 1: If $S \geq 2^{n-1} + 1$, then

- (i) Payoff set $(X_{p_m}) = A_m$ for $m < S$, and
- (ii) Payoff set $(X_{p_m}) = A_S \cup A_{S+1} \cup L \cup A_{2^n}$.

Case 2: If $S \leq 2^{n-1}$, then

- (i) Payoff set $(X_{p_m}) = A_m$ for $m \leq S - 1$, and
- (ii) Payoff set $(X_{p_{S-1}}) = A_{S-1} \cup A_S \cup L \cup A_{2^{n-1}}$, and
- (iii) Payoff set $(X_{p_m}) = A_{2^{n-1}+1} \cup A_{2^{n-1}+2} \cup L \cup A_{2^n}$

In either case, payoff set (X_{p_1}) will be either

- (a) a subset of $A_1 \cup L \cup A_{2^{n-1}} \subseteq \{\text{plays } f \mid \text{player } X_{j(1)} \text{ plays } f(w_{j(1)}) = a\}$

or

- (b) a subset of $A_{2^{n-1}+1} \cup L \cup A_{2^n} \subseteq \{\text{plays } f \mid \text{player } X_{j(1)} \text{ plays } f(w_{j(1)}) \neq a\}$.⁵

⁵ **Claim 1.** $\forall \alpha < \kappa$:

Payoff set $(X_\alpha) \subseteq A_1 \cup A_2 \cup L \cup A_{2^{n-1}}$, or Payoff set $(X_\alpha) \subseteq A_{2^{n-1}+1} \cup A_{2^{n-1}+2} \cup L \cup A_{2^n}$.

Proof. Pick an $\alpha < \kappa$.

Claim 1 clearly holds if payoff set $(X_\alpha) = \emptyset$. Also Claim 1 holds for Case 1-(i) and Case 2-(ii) since payoff set $(X_{p_m}) = A_m$.

In Case 1-(ii), payoff set $(X_{p_S}) =_{\text{df}} A_S \cup L \cup A_{2^n} \subseteq A_{2^{n-1}+1} \cup L \cup A_{2^n}$ for $S \geq 2^{n-1} + 1$.

In Case 2-(ii), payoff set $(X_{p_{S-1}}) =_{\text{df}} A_{S-1} \cup L \cup A_{2^{n-1}} \subseteq A_1 \cup A_2 \cup L \cup A_{2^{n-1}}$ for $S \leq 2^{n-1}$.

In Case 2-(iii), payoff set $(X_{p_S}) =_{\text{df}} A_{2^{n-1}+1} \cup A_{2^{n-1}+2} \cup L \cup A_{2^n}$. , (Claim 1)

Since $A_1 \cup A_2 \cup L \cup A_{2^{n-1}} \subseteq \{\text{plays } f \mid f(w_{j(1)}) = a\}$ and

$A_{2^{n-1}+1} \cup A_{2^{n-1}+2} \cup L \cup A_{2^n} \subseteq \{\text{plays } f \mid f(w_{j(1)}) \neq a\}$, by Claim 1 we get:

Claim 2. $\forall \alpha < \kappa$:

Payoff set $(X_\alpha) \subseteq \{\text{plays } f \mid f(w_{j(1)}) = a\}$ or Payoff set $(X_\alpha) \subseteq \{\text{plays } f \mid f(w_{j(1)}) \neq a\}$.

Therefore, if $p_1 \neq j(1)$, player $X_{j(1)}$ can make player X_{p_1} lose by playing an appropriate value for $f(w_{j(1)})$: $f(w_{j(1)}) \neq a$ in case (a) or $f(w_{j(1)}) = a$ in case (b). If player $X_{j(1)}$ has nonempty payoff, then $p_1 = j(1)$ (by our definition of p_1),

$$\text{payoff set}(X_{j(1)}) = \text{payoff set}(X_{p_1}) =$$

$$A_1 \subseteq \{\text{plays } f \mid \text{each } R_n (2 \leq h \leq n) \text{ is } =\}.$$

Therefore, $\text{payoff set}(X_{j(1)}) \subseteq \{\text{plays } f \mid f(w_{j(2)}) = f(w_{j(3)}) = \dots = f(w_{j(n)}) = a\}$ so that any player $X_{j(h)}$ for $2 \leq h \leq n$ can make player $X_{j(1)}$ lose by playing anything different from a . Players other than X_{p_1} for $1 \leq i \leq S$ have empty payoff and therefore always lose. Consequently, no player can have a winning strategy. For readers who require more details we prove the following:

Claim. The game G is not determined.

Pick an arbitrary player X_i where $0 \leq i \leq \kappa$ and pick an arbitrary strategy s_i for player X_i . We will show s_i is not a winning strategy, i.e. there exists a play $\overset{1}{y}$ according to s_i such that $\overset{r}{y} \notin \text{payoff set}(X_i)$.

Case 1: Let $i \neq p_1$ for all $1 \leq i \leq S$, i.e. $\text{payoff set}(X_i) = \emptyset$.

Let $\overset{1}{y}$ be a play according to s_i . Since X_i has empty payoff, $\overset{1}{y}$ is a loss for player X_i . Therefore, s_i is not a winning strategy.

Case 2: Let $i = p_1$ for some $1 \leq i \leq S$ and $i \neq j(1)$. (Recall $S \geq 3$.)

Recall $X_{p_1}, X_{p_2}, X_{p_3}, \dots, X_{p_s}$ are the players with nonempty payoff. Since $i \neq j(1)$, player X_{p_i} is different from player $X_{j(1)}$. By the definition of the payoff sets,

$$\text{payoff}(X_{p_i}) \subseteq \{\text{plays } f \mid f(w_{j(1)}) = a\} \text{ or } \text{payoff}(X_{p_i}) \subseteq \{\text{plays } f \mid f(w_{j(1)}) \neq a\}.$$

First, suppose payoff set $(X_{p_i}) \subseteq \{\text{plays } f \mid f(w_{j(1)}) = a\}$. Let $\overset{1}{y}$ be a play according to s_{p_i} in which player $X_{j(1)}$ plays $f(w_{j(1)}) \neq a$. Then $\overset{1}{y} \notin \text{payoff set}(X_{p_i})$ and therefore $\overset{1}{y}$ is a loss for player X_{p_i} . Therefore, s_{p_i} is not a winning strategy for player X_{p_i} .

Next, suppose payoff set $(X_{p_i}) \subseteq \{\text{plays } f \mid f(w_{j(1)}) \neq a\}$. Let $\overset{1}{y}$ be a play according to s_{p_i} in which player $X_{j(1)}$ plays $f(w_{j(1)}) = a$. Then $\overset{1}{y} \notin \text{payoff set}(X_{p_i})$ and therefore $\overset{1}{y}$ is a loss for player X_{p_i} . Therefore, s_{p_i} is not a winning strategy for player X_{p_i} .

Consequently, in both cases, since $i = p_i$, s_i is not a winning strategy.

Case 3: Let $i = p_1$ for some $1 \leq l \leq S$ and $i = j(1)$. (Recall $S \geq 3$.)

Since $j(1) = i = p_1$ for some $1 \leq l \leq S$, $l = 1$ and $j(1) = i = p_1$ by the definition of the p_1 . Therefore player $X_{j(1)}$ (i.e. X_{p_1}) has payoff set A_1 .

Let $\overset{1}{y}$ be a play according to $s_{j(1)}$ in which any player $X_{j(h)}$ for $2 \leq h \leq n$ plays $f(w_{j(h)}) \neq a$. Since

$$A_1 \subseteq \{\text{plays } f \mid f(w_{j(2)}) = f(w_{j(3)}) = f(w_{j(4)}) = \dots = f(w_{j(n)}) = a\},$$

then $\overset{1}{y} \notin \text{payoff set}(X_{j(1)})$ and $\overset{1}{y}$ is a loss for player $X_{j(1)}$. Therefore, $s_{j(1)}$ is not a winning strategy (i.e. s_i is not a winning strategy).

In summary, s_i is not a winning strategy. Since s_i is an arbitrary strategy for player X_i and i was arbitrary, the game is not determined. , (Theorem 1.6)

CHAPTER 2

DETERMINANCY OF A THREE-PLAYER BIASED

GAME: A WARM-UP

Chapter 2 consists of two subsections, Section 2.1 and Section 2.2. In Section 2.1 we review standard material that will be used later in Section 2.2.

In Section 2.1 we review a proof of the determinacy of two-player open games. The proof we review involves ordinals of a position. We shall need the notion of an ordinal of a position to define our three-player biased open games. We shall also need the standard techniques involving ordinals of a position to prove determinacy for certain biased games introduced in Section 2.2 and Chapter 3.

In Section 2.2, we make our first attempt at proving the determinacy of certain three-player biased open games. We generalize our determinacy result of Section 2.2 in Chapter 3. A main difference in the games in Section 2.2 and Chapter 3 is that one of the players has empty payoff set. In Section 2.2 we introduce the concept of a player “not helping” another player. In this game the player with the empty payoff set is not allowed to help the player with the open payoff set.

Section 2.1

Suppose there is a set of positions that player I “should find desirable” to reach, whereas player II finds these positions “undesirable”.¹ We shall show in Theorem 2.1 that there is a particular property about the empty position $\langle \rangle$ such that:

- Player I has a strategy to reach one of the desirable positions when this property holds for $\langle \rangle$.
- Player II has a strategy to avoid all of these positions desirable for player I when this property fails for $\langle \rangle$.

We now build up to defining this property.

Definition 2.1. Definition of ordinals of a position in two-player games.

Let D be a collection of positions.² We assign certain positions an ordinal value with respect to D as follows:

- (i) A position $\overset{\text{I}}{p}$ has $\text{ORD}_D 0$ (ordinal zero with respect to D) iff $\overset{\text{I}}{p}$ extends some position in D (possibly $\overset{\text{I}}{p} \in D$),
- (ii) $\overset{\text{I}}{p} = (a_0, a_1, \mathbf{K}, a_{2n-2}, a_{2n-1})$ has $\text{ORD}_D \beta$ iff

$$\exists f(2n) = a_{2n} \left[\overset{\text{I}}{p} \right] (f(2n)) \text{ has } \text{ORD}_D < \beta,$$

¹ Such a set of positions occurs when we consider open games. If A is open, then there exists positions $\overset{\text{I}}{p}_1, \overset{\text{I}}{p}_2, \overset{\text{I}}{p}_3, \mathbf{K}$ such that:

$$f \in A \text{ iff } \exists i \overset{\text{I}}{p}_i \subseteq f.$$

Player I wins the open game A exactly in the case in which one of the $\overset{\text{I}}{p}_i$'s has been reached. If both players wish to win the game A , then player I would find reaching one of the $\overset{\text{I}}{p}_i$'s desirable, whereas player II would find this undesirable (as reaching such a $\overset{\text{I}}{p}_i$ results in a win for I and a loss for II). Another example of this occurs when we consider the collection of positions $\overset{\text{I}}{p}_i$ at which player I has a winning strategy for the remainder of the game B (even if B is not open).

² Possibly these are positions player I finds desirable.

(iii) $\overset{\text{I}}{p} = (a_0, a_1, \mathbf{K}, a_{2n-1}, a_{2n})$ has $\text{ORD}_D \beta$ iff

$$\forall f(2n+1) = a_{2n+1} \left[\overset{\text{I}}{p} \right] (f(2n+1)) \text{ has } \text{ORD}_D \leq \beta \text{.}$$

Let $\overset{\text{I}}{p} \in \text{ORD}_D \beta$ iff $\overset{\text{I}}{p}$ has ordinal β with respect to D . Let $\overset{\text{I}}{p} \in \text{ORD}_D$ iff $\overset{\text{I}}{p} \in \text{ORD}_D \beta$

for some ordinal β . Also let $\text{ORD}_D(\overset{\text{I}}{p}) \downarrow$ (read: ordinal of $\overset{\text{I}}{p}$ converges) iff $\overset{\text{I}}{p} \in \text{ORD}_D$.

Let $\text{ORD}_D(\overset{\text{I}}{p}) \uparrow$ (read: ordinal of $\overset{\text{I}}{p}$ diverges) iff $\overset{\text{I}}{p} \notin \text{ORD}_D$.

Typically we drop the D in ORD_D and $\text{ORD}_D \beta$, as D will be clear from the context.

The following is clear from the definition of $\text{ORD}_D 0$.

Fact 2.2. If the $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{m-1}, a_m) \downarrow$ and $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{m-1}, a_m) = 0$, then

any position that extends $(a_0, a_1, \mathbf{K}, a_{m-1}, a_m)$ also has $\text{ORD}_D 0$. (Fact 2.2)

Theorem 2.3. Let D be a collection of positions. If $\langle \rangle \in \text{ORD}_D$, then player I has a strategy to reach a position (in any two-player game) which has $\text{ORD}_D 0$ and therefore a position in D is reached when player I follows this strategy. If $\langle \rangle \notin \text{ORD}_D$, then player II has a strategy to keep all positions from having an ordinal and in particular from being in D so that any play according to this strategy is not in the open set generated by D .

Proof. Let D be a collection of positions in the game:

$$G: \begin{array}{ccccccc} \text{I} & a_0 & & a_2 & & a_{2n} & \\ & & & & & & \\ \text{II} & & a_1 & & a_3 & & a_{2n+1} \end{array}$$

Case 1: Suppose $\langle \rangle \in \text{ORD}_D$.

In this case we show there exists a strategy for player I so that a position which has $\text{ORD}_D 0$ is reached. Define a strategy for player I as follows:³

$$\text{Let } s(a_0, a_1, \mathbf{K}, a_{2n-2}, a_{2n-1}) = \begin{cases} \text{the } \mu a_{2n} \text{ if } \exists x = a_{2n} \\ \left[\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-1}, x) \downarrow \right. \\ \text{and } \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-2}, a_{2n-1}) \downarrow, \\ \text{and } \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-1}, x) \\ \left. < \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-2}, a_{2n-1}) \right], \\ 8 \text{ otherwise, i.e.} \\ \left[\forall x \left[\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-2}, a_{2n-1}) \uparrow, \right. \right. \\ \text{or } \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-1}, x) \uparrow, \\ \text{or } \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-2}, a_{2n-1}) \downarrow \\ \text{and } \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-1}, x) \downarrow \\ \text{and } \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-2}, a_{2n-1}) \\ \left. \left. \leq \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n-1}, x) \right] \right]. \end{cases}$$

Let $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ be a legal (infinite) play according to s . We shall show

$$\exists n \left[\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow \text{ and } \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) = 0 \right].$$

Claim 1. $\forall n \in \omega \left[\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow \text{ and either}$

$$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) = 0 \text{ or}$$

$$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \mathbf{R}_n \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \text{ where}$$

³ In our definition of s below, we abbreviate “the least i ” by “ μi ”. It is standard in mathematical logic to use μ for “the least”.

$$\mathbf{R}_n = \begin{cases} < \text{ if } n \text{ is even,} \\ \leq \text{ if } n \text{ is odd} \end{cases}.$$

Considering the two cases “ n is even” or “ n is odd”, we show by induction that Claim 1 holds. Fix $n \in \omega$. We first note that $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \downarrow$: this follows by the Induction Hypothesis when $n \geq 1$ and otherwise by the assumption of Case 1 that $\langle \rangle \in \text{ORD}_D$. Let $\gamma = \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1})$.

Sub-case 1.1. n is even, i.e. $n = 2k$ for some $k \geq 0$.

Show that $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \downarrow$ and either

$$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) = 0 \text{ or}$$

$$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) < \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-2}, a_{2k-1}).$$

Recall $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-2}, a_{2k-1}) \downarrow$ and $\gamma = \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-2}, a_{2k-1})$. If $\gamma = 0$, then by Fact 2.2 $\forall x (a_0, a_1, \mathbf{K}, a_{2k-1}, x)$ has ordinal zero; in particular, $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \downarrow$ and is zero. If $\gamma > 0$, then by the definition of ORD_D ,

$$\exists x \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, x) \downarrow \text{ and}$$

$$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, x) < \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-2}, a_{2k-1}).$$

By the definition of s, $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \downarrow$ and

$< \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-2}, a_{2k-1})$ since $(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k})$ is according to s.

Sub-case 1.2. n is odd, i.e. $n = 2k + 1$ for some $k \geq 0$.

Show that the $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, a_{2k+1}) \downarrow$ and either

$$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, a_{2k+1}) = 0 \text{ or}$$

$$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, a_{2k+1}) \leq \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}).$$

Recall $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \downarrow$ and $\gamma = \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k})$.

By the definition of ORD_D ,

$$\forall x \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, x) \downarrow \text{ and is } \leq \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) = \gamma,$$

regardless of whether $\gamma = 0$ or $\gamma > 0$. In particular $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, a_{2k+1}) \downarrow$ and is $\leq \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k})$.

So by Sub-cases 1.1 and 1.2 we have $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow$ and Claim 1 holds for our fixed n . Thus, Claim 1 has been shown by induction. , (Claim 1)

By Claim 1, if $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \neq 0$, then

$$\begin{aligned} & \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \\ & < \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-2}, a_{2k-1}) \leq \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-3}, a_{2k-2}). \end{aligned}$$

Hence if $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \neq 0$, then $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-3}, a_{2k-2}) \neq 0$ and

$$\begin{aligned} & \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) < \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-3}, a_{2k-2}) \\ & < \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-3}, a_{2k-4}) < L < \text{ORD}_D(a_0). \end{aligned}$$

Therefore if $\forall k \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \neq 0$, then

$$\begin{aligned} & \text{ORD}_D(a_0) > \text{ORD}_D(a_0, a_1, a_2) > L \\ & > \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) > \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k+1}, a_{2k+2}) > L \end{aligned}$$

which gives an infinite decreasing sequence of ordinals. Since a decreasing sequence of ordinals must be finite, $\exists k \text{ ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) = 0$. By the definition of $\text{ORD}_D 0$

$$\exists i \leq 2k (a_0, a_1, \mathbf{K}, a_{i-1}, a_i) \in D. \quad , \quad (\text{Case 1})$$

Case 2: Suppose $\langle \rangle \notin \text{ORD}_D$.

In this case we show there exists a strategy for player II such that all positions do not have an ordinal. Define a strategy for player II as follows:

$$\text{Let } s(a_0, a_1, \mathbf{K}, a_{2n-1}, a_{2n}) = \begin{cases} \text{the } \mu a_{2n+1} \text{ such that} \\ \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2n}, a_{2n+1}) \uparrow, \\ 8 \text{ otherwise,} \\ \text{if there is no such } a_{2n+1}. \end{cases}$$

Let $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ be a legal (infinite) play according to s .

Claim 2. $\forall n \in \omega (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin \text{ORD}_D$.

We show by induction that Claim 2 holds. So fix $n \in \omega$. Note that

$$(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \notin \text{ORD}_D \quad (\mathfrak{A})$$

as this follows by the Induction Hypothesis when $n-1 \in \mathbb{N}$ and otherwise by the assumption of Case 2 that $\langle \rangle \notin \text{ORD}_D$. From this, we are to show

$(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin \text{ORD}_D$ (follows from (\mathfrak{A})). Instead we shall show the

contrapositive: we assume $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \in \text{ORD}_D$ and show

$(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \in \text{ORD}_D$. Let $\gamma = \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n)$.

Sub-case 2.1. n is even, i.e. $n = 2k$ for some $k \geq 0$.

Since $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) = \gamma$, by the definition of ORD_D ,

$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-2}, a_{2k-1}) \downarrow$ and equals $\gamma+1$.

Sub-case 2.2. n is odd, i.e. $n = 2k+1$ for some $k \geq 0$.

Since $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, a_{2k+1}) \downarrow$ and $(a_0, a_1, \mathbf{K}, a_{2k}, a_{2k+1})$ is played

according to s , $\forall x \text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, x) \downarrow$. Let $\delta = \sup_x [\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, x)]$.

Since $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k}, x) \leq \delta$ for every x , by the definition of ORD_D ,

$\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{2k-1}, a_{2k}) \downarrow$ and equals δ .

So by Sub-cases 1.1 and 2.2 we have shown $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \downarrow$ follows from $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow$ for our fixed n . Since $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \uparrow$, $\text{ORD}_D(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \uparrow$. Thus, Claim 2 has been shown by induction. , (Claim 2)

Consequently, if $\langle \rangle \notin \text{ORD}_D$ and $\overset{\mathbf{r}}{y} = (a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ is a play according to s , then $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin \text{ORD}_D$ and in particular $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin D$ so that the play $\overset{\mathbf{l}}{y}$ is not in the open set generated by D . , (Case 2 and Theorem 2.3)

In Corollary 2.7 below, we note that the determinacy of certain types of games, open two-player games (defined below in Definition 2.6), immediately follows from Theorem 2.3.

Definition 2.4. Definition of an open set.

Let D be a set of positions. Define $O(D) =_{\text{df}} \{\text{play } f \mid \exists \text{ position } \overset{\text{r}}{p} \in D \text{ such that } \overset{\text{r}}{p} \subseteq f\}$,
i.e. $O(D)$ consists exactly of those plays which extend some position in D . A set A of
plays is called *open* exactly when $A = O(D)$ for some set D of positions; in this case, we
say that D *generates* the open set A .

Definition 2.5. Definition of a game being open for a player.

Let G be a game with $\kappa \geq 2$ players⁴ and let X be one of the players. We say that G
is *open* for player X iff the payoff set for player X is open, i.e. \exists set D of positions such
that the payoff set for player X consists exactly of those plays which extend some
position in D .

Definition 2.6. Definition of an open game.

A two-player game G is open iff it is open for player I, i.e. the payoff set A for player
I is open.

In Theorem 2.3, (above) we showed that either player I has a strategy s_D to get into
 D , or player II has a strategy s_D to stay out of D (depending on whether $\langle \rangle \in \text{ORD}_D$ or
 $\langle \rangle \notin \text{ORD}_D$). If A is the open set generated by D , then the strategy s_D naturally leads to a
winning strategy for the game G , so that the following holds:

⁴ $\kappa \geq 2$ is an ordinal.

Corollary 2.7. Every two-player open game is determined. Moreover:

- (i) Player I has a winning strategy for a two-player open game A exactly when $\langle \rangle \in \text{ORD}_D$, for some (any) D that generates A .
- (ii) Player II has a winning strategy for a two-player open game A exactly when $\langle \rangle \notin \text{ORD}_D$, for some (any) D that generates A .⁵

Proof. Let G be an open game, i.e. there exist finite sequences $\overset{\text{I}}{p}_1, \overset{\text{I}}{p}_2, \overset{\text{I}}{p}_3, \mathbf{K}$ such that:

$$f \in A \text{ iff } \exists i \overset{\text{I}}{p}_i \subseteq f. \quad (*)$$

Then $\{\overset{\text{I}}{p}_{i+1} \mid i \in \omega\}$ generates A . Let D be any set that generates A . We show that one of the players has a winning strategy for the game G by considering the cases $\langle \rangle \in \text{ORD}_D$ and $\langle \rangle \notin \text{ORD}_D$.

Case 1: Suppose $\langle \rangle \in \text{ORD}_D$.

In this case, we show that player I has a winning strategy for the game G . By Theorem 2.3, player I has a strategy s to reach a position $\overset{\text{I}}{p}$ with ordinal zero. Let σ be the strategy in which player I plays according to s until he reaches such a $\overset{\text{I}}{p}$, and then he plays randomly. If $\overset{\text{I}}{y} = (y_0, y_1, y_2, \mathbf{K})$ is a play according to σ , then:

$$\exists n (y_0, y_1, \mathbf{K}, y_{n-1}, y_n) \text{ has ordinal zero.}$$

By the definition of $\text{ORD}_D 0$, $\exists i \leq n (y_0, y_1, y_2, \mathbf{L}, y_{i-1}, y_i) \in D$. By (*) $\overset{\text{I}}{y} \in A$ and

⁵ If $\langle \rangle \in \text{ORD}_D$ (respectively $\langle \rangle \notin \text{ORD}_D$) for some D that generates A , then $\langle \rangle \in \text{ORD}_D$ (respectively $\langle \rangle \notin \text{ORD}_D$) for all D that generate A .

therefore is a win for player I. Consequently, σ is a winning strategy for player I if

$$\langle \rangle \in \text{ORD}_D.$$

Case 2: Suppose $\langle \rangle \notin \text{ORD}_D$.

In this case, we show that player II has a winning strategy for the game G . By Theorem 2.3, player II has a strategy s to keep all positions according to s from being in D . Therefore, if $\overset{\text{I}}{y} = (y_0, y_1, y_2, \dots)$ is a play according to s , then

$$\forall n (y_0, y_1, y_2, \dots, y_{n-1}, y_n) \notin D; \text{ therefore by } (*), \overset{\text{I}}{y} \notin A \text{ and is a win for player II.}$$

Consequently, if $\langle \rangle \notin \text{ORD}_D$, then s is a winning strategy for player II.

Thus, either player I or player II has a winning strategy in the game G so that G is determined. , (Corollary 2.7)

We lastly note that if $\langle \rangle \in \text{ORD}_D$ (respectively $\langle \rangle \notin \text{ORD}_D$) for some D that generates A , then $\langle \rangle \in \text{ORD}_D$ (respectively $\langle \rangle \notin \text{ORD}_D$) for all D that generates A . Otherwise, $\langle \rangle \in \text{ORD}_D$ and $\langle \rangle \notin \text{ORD}_E$ for some D and E that generate A . Since $\langle \rangle \in \text{ORD}_D$, player I has a winning strategy for the game A , and since $\langle \rangle \notin \text{ORD}_E$, player II has a winning strategy for the game G . Playing I and II's strategy against one another leads to a play that is both in and outside of A , a contradiction.

Section 2.2

Consider a three-player game with an empty payoff set assigned to player III and an open payoff set assigned to player I. We know such games are not determined.

However, with appropriate restrictions on player III we shall show such games with these restrictions are determined. The appropriate restrictions on player III require that player III “not help” player I. The definition of “not helping” is somewhat technical and will require the definition of ordinals of a position. This will be defined shortly.

We first define $\text{ORD}_1^{3,D}$, the ordinals of positions for player I in three-player games with respect to D .

Definition 2.8. Definition of ordinals of a position for three-player games.

Let D be a collection of positions. We assign certain positions an ordinal value with respect to D as follows:

- (i) $\overset{\mathbf{r}}{p} \in \text{ORD}_1^{3,D} 0$ iff $\exists \overset{\mathbf{r}}{q} \in D$ ($\overset{\mathbf{r}}{q} \subseteq \overset{\mathbf{r}}{p}$),
- (ii) $\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{3n-2}, a_{3n-1}) \in \text{ORD}_1^{3,D} \beta$ iff
 $\exists f(3n) = a_{3n} \left[\overset{\mathbf{r}}{p} \right] (f(3n))$ has $\text{ORD}_1^{3,D} < \beta$,
- (iii) $\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) \in \text{ORD}_1^{3,D} \beta$ iff
 $\forall f(3n+1) = a_{3n+1} \left[\overset{\mathbf{r}}{p} \right] (f(3n+1))$ has $\text{ORD}_1^{3,D} \leq \beta$.
- (iv) $\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \in \text{ORD}_1^{3,D} \beta$ iff
 $\forall f(3n+2) = a_{3n+2} \left[\overset{\mathbf{r}}{p} \right] (f(3n+2))$ has $\text{ORD}_1^{3,D} \leq \beta$

Let $\overset{\Gamma}{p} \in \text{ORD}_I^{3,D} \beta$ iff $\overset{1}{p}$ has ordinal β with respect to player I in a three-player game generated by the set D . Let $\overset{\Gamma}{p} \in \text{ORD}_I^{3,D}$ iff $\overset{\Gamma}{p} \in \text{ORD}_I^{3,D} \beta$ for some ordinal β . Also let $\text{ORD}_I^{3,D}(\overset{\Gamma}{p}) \downarrow$ (read: ordinal of $\overset{1}{p}$ converges) iff $(\overset{\Gamma}{p}) \in \text{ORD}_I^{3,D}$. Let $\text{ORD}_I^{3,D}(\overset{\Gamma}{p}) \uparrow$ (read: ordinal of $\overset{1}{p}$ diverges) iff $(\overset{\Gamma}{p}) \notin \text{ORD}_I^{3,D}$.

Typically we drop the 3 and D in the superscript and the I in the subscript as these are clear from the context.

The following is clear from the definition of $\text{ORD}_I^{3,D} 0$.

Fact 2.9. If the $\text{ORD}_I^{3,D}(a_0, a_1, K, a_{m-1}, a_m) \downarrow$ and $\text{ORD}_I^{3,D}(a_0, a_1, K, a_{m-1}, a_m) = 0$, then any position that extends $(a_0, a_1, K, a_{m-1}, a_m)$ also has $\text{ORD}_I^{3,D} 0$. (Fact 2.9)

Next we define a player “not helping” player I in three-player games in which player I has an open payoff set.

Definition 2.10. Definition of player III not helping player I with respect to D , denoted

by $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$.

A three-player game G satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ (read: III doesn't help I with respect to D) iff the following holds:

If $\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \notin \text{ORD}_I^{3,D}$ and if $\exists x_{3n+2}$ such that $\overset{\mathbf{r}}{p}'(x_{3n+2}) \notin \text{ORD}_I^{3,D}$, then in the game G player III can only play a_{3n+2} such that $\overset{\mathbf{r}}{p}'(a_{3n+2}) \notin \text{ORD}_I^{3,D}$.⁶,

If G satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \notin \text{ORD}_I^{3,D}$, then player III must play a move a_{3n+2} that keeps the position $\overset{\mathbf{r}}{p}'(a_{3n+2})$ out of $\text{ORD}_I^{3,D}$ whenever possible, i.e. whenever $\exists x_{3n+2} \overset{\mathbf{r}}{p}'(x_{3n+2}) \notin \text{ORD}_I^{3,D}$. We next prove that such a move a_{3n+2} always exists if $\overset{\mathbf{r}}{p} \notin \text{ORD}_I^{3,D}$ so that we can drop the “whenever possible” condition from Definition 2.10.

Proposition 2.11. Let D be a collection of positions. If

$\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \notin \text{ORD}_I^{3,D}$, then $\exists x_{3n+2}$ such that $\overset{\mathbf{r}}{p}'(x_{3n+2}) \notin \text{ORD}_I^{3,D}$.

Proof. Proof by contraposition. Suppose $\forall x \text{ORD}_I^{3,D}(a_0, a_1, \mathbf{K}, a_{3n+1}, x) \downarrow$. Let

$\gamma = \sup_x [\text{ORD}_I^{3,D}(a_0, a_1, \mathbf{K}, a_{3n+1}, x)]$. By the definition of $\text{ORD}_I^{3,D}$,

$\text{ORD}_I^{3,D}(a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \downarrow$ and is $= \gamma$. , (Proposition 2.11)

Therefore, if $(a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \notin \text{ORD}_I^{3,D}$, then there is an x_{3n+2} such that $(a_0, a_1, \mathbf{K}, a_{3n+1}, x_{3n+2}) \notin \text{ORD}_I^{3,D}$. Hence, in any game that satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$,

⁶ In Chapter 3 we give the obvious generalizations of Definitions 2.8 and 2.10.

whenever $(a_0, a_1, K, a_{3n}, a_{3n+1}) \notin \text{ORD}_I^{3,D}$, player III actually must play a_{3n+2} such that $(a_0, a_1, K, a_{3n+1}, a_{3n+2}) \notin \text{ORD}_I^{3,D}$.

Theorem 2.12. Let D be a collection of positions. If $\langle \rangle \in \text{ORD}_I^{3,D}$, then player I has a strategy to reach a position which has $\text{ORD}_I^{3,D} 0$ in every three-player game and therefore a position in D is reached when player I follows this strategy. If $\langle \rangle \notin \text{ORD}_I^{3,D}$ and G is a three-player game that satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, then player II has a strategy to keep all positions from having an $\text{ORD}_I^{3,D}$ in the game G and in particular from being in D so that any play according to this strategy is not in the open set generated by D .

Proof. Let D be a collection of positions in the game:

$$\begin{array}{ccccccc}
 & \text{I} & a_0 & & a_3 & & a_{3n} \\
 G: & \text{II} & & a_1 & & a_4 & \text{L} & & a_{3n+1} & & \text{L} & . \\
 & \text{III} & & & a_2 & & & a_5 & & & & a_{3n+2}
 \end{array}$$

Case 1: Suppose $\langle \rangle \in \text{ORD}_I^{3,D}$.

Show there exists a strategy for player I so that a position which has $\text{ORD} 0$ is reached.⁷ We construct a strategy here for player I analogous to our construction (in the two-player game) for player I in Theorem 2.3. Namely, player I plays “to strictly lower the ordinal values” of positions until a position with ordinal value zero is reached. Define a strategy for player I as follows:

⁷ We suppress the superscripts 3 and D which indicate this is a three-player game with respect to D ; and the subscript I which indicates player I has the assigned ordinals of position for this game.

$$\text{Let } s(a_0, a_1, \mathbf{K}, a_{3n-2}, a_{3n-1}) = \begin{cases} \text{the } \mu a_{3n} \text{ if } \exists x = a_{3n} \\ \left[\text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-1}, x) \downarrow \right. \\ \text{and } \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-2}, a_{3n-1}) \downarrow, \\ \text{and } \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-1}, x) \\ \left. < \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-2}, a_{3n-1}) \right], \\ \\ 8 \text{ otherwise, i.e.} \\ \forall x \left[\text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-1}, x) \uparrow, \right. \\ \text{or } \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-2}, a_{3n-1}) \uparrow, \\ \text{or } \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-2}, a_{3n-1}) \downarrow \\ \text{and } \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-1}, x) \downarrow \\ \text{and } \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-2}, a_{3n-1}) \\ \left. \leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n-1}, x) \right]. \end{cases}$$

Let $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ be a legal play according to s . We will show

$$\exists n \left[\text{ORD}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow \text{ and } \text{ORD}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) = 0 \right].$$

Claim 1. $\forall n \in \omega \left[\text{ORD}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow \text{ and either}$

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) = 0 \text{ or}$$

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \mathbf{R}_n \text{ORD}(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \text{ where}$$

$$\mathbf{R}_n = \begin{cases} < \text{ if } n = 3l, \text{ for some } l \\ \leq \text{ if } n \neq 3l, \text{ for all } l \end{cases}.$$

Considering the cases “ $n = 3l$ ”, “ $n = 3l + 1$ ”, “ $n = 3l + 2$ ” we show by induction that

Claim 1 holds. Fix $n \in \omega$. We first note that $\text{ORD}(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \downarrow$: this follows

by the Induction Hypothesis when $n \geq 1$ and otherwise by the assumption of Case 1 that

$$\langle \rangle \in \text{ORD}. \text{ Let } \gamma = \text{ORD}(a_0, a_1, a_2, \mathbf{L}, a_{n-2}, a_{n-1}).$$

Sub-case 1.1. $n = 3k$ for some $k \geq 0$.

Show that $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \downarrow$ and either

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) = 0 \text{ or}$$

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) < \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}).$$

Recall $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \downarrow$ and $\gamma = \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1})$. If $\gamma = 0$, then by Fact 2.9 $\forall x (a_0, a_1, \mathbf{K}, a_{3k-1}, x)$ has ordinal zero; in particular,

$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \downarrow$ and is zero. If $\gamma > 0$, then by the definition of ORD ,

$$\exists x \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, x) \downarrow \text{ and}$$

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, x) < \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}).$$

By the definition of s , $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \downarrow$ and $< \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1})$ since $(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k})$ is according to s .

Sub-case 1.2. $n = 3k + 1$ for some $k \geq 0$.

Show that the $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \downarrow$ and either

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) = 0 \text{ or}$$

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}).$$

Recall $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \downarrow$ and $\gamma = \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k})$. By the definition of ORD ,

$$\forall x \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, x) \downarrow \text{ and is } \leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) = \gamma,$$

regardless of whether $\gamma = 0$ or $\gamma > 0$. In particular $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \downarrow$ and is $\leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k})$.

Sub-case 1.3. $n = 3k + 2$ for some $k \geq 0$.

By the “same” argument just presented in Sub-case (2), we show that

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) \downarrow \text{ and either}$$

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) = 0 \text{ or}$$

$$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) \leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}).$$

Recall $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \downarrow$ and $\gamma = \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1})$. By the definition of ORD,

$$\forall x \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k+1}, x) \downarrow \text{ and is } \leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) = \gamma$$

regardless of whether $\gamma = 0$ or $\gamma > 0$. In particular $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) \downarrow$ and is $\leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1})$.

So by Sub-cases 1.1, 1.2, and 1.3, we have $\text{ORD}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow$ and Claim 1 holds for our fixed n . Thus, Claim 1 has been shown by induction. , (Claim 1)

By Claim 1, if $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \neq 0$, then

$$\begin{aligned} & \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) < \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \\ & \leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-3}, a_{3k-2}) \leq \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-4}, a_{3k-3}). \end{aligned}$$

Hence if $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \neq 0$, then

$$\begin{aligned} & \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-4}, a_{3k-3}) \neq 0 \text{ and} \\ & \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) < \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-4}, a_{3k-3}) \\ & < \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-7}, a_{3k-6}) < L < \text{ORD}(a_0). \end{aligned}$$

Therefore if $\forall k \text{ ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \neq 0$, then $\text{ORD}(a_0) > \text{ORD}(a_0, a_1, a_2, a_3)$

$> L > \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) > \text{ORD}(a_0, a_1, \mathbf{K}, a_{3k+2}, a_{3k+3}) > L$ which gives an

infinite decreasing sequence of ordinals. Since a decreasing sequence of ordinals must be

finite, $\exists k \text{ ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) = 0$. By the definition of ordinal zero,

$\exists i \leq 3k (a_0, a_1, \mathbf{K}, a_{i-1}, a_i) \in D$. , (Case 1)

Case 2: Suppose $\langle \rangle \notin \text{ORD}$.

In this case we show there exists a strategy for player II such that all positions do not have an ordinal. Define a strategy for player II as follows:

$$\text{Let } s(a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) = \begin{cases} \text{the } \mu a_{3n+1} \text{ such that} \\ \text{ORD}(a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \uparrow, \\ 8 \text{ otherwise,} \\ \text{if there is no such } a_{3n+1}. \end{cases}$$

Let $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ be a legal, infinite play according to s .

Claim 2. $\forall n \in \omega (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin \text{ORD}$.

We show by induction that Claim 2 holds. So fix $n \in \omega$. Note that

$$(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \notin \text{ORD} \quad (\spadesuit)$$

as this follows by the Induction Hypothesis when $n-1 \in \mathbb{Y}$ and otherwise by the

assumption of Case 2 that $\langle \rangle \notin \text{ORD}$. From this, we are to show

$(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin \text{ORD}$ follows from (\spadesuit) . In the first two sub-cases below, $n = 3k$

or $n = 3k + 1$, we instead shall show the contrapositive: we assume

$(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \in \text{ORD}$, let $\gamma = \text{ORD}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n)$, and show

$(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \in \text{ORD}$.

Sub-case 2.1. $n = 3k$ for some $k \geq 0$.

Since $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) = \gamma$, by the definition of ORD ,

$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \downarrow$ and equals $\gamma + 1$.

Sub-case 2.2. $n = 3k + 1$ for some $k \geq 0$.

Since $\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \downarrow$ and $(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k+1})$ is played

according to s , $\forall x (a_0, a_1, \mathbf{K}, a_{3k}, x) \downarrow$. Let $\delta = \sup_x [\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, x)]$. Since

$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, x) \leq \delta$ for every x , by the definition of ORD ,

$\text{ORD}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \downarrow$ and equals δ .

Sub-case 2.3. $n = 3k + 2$ for some $k \geq 0$.

Here we directly show $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin \text{ORD}$ follows from (\boxplus) .

Recall (\boxplus) , i.e. $(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \notin \text{ORD}$. Since $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, by Proposition

2.11, player III plays a legal move a_{3n+2} such that $(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) \notin \text{ORD}$.

So by Sub-cases 2.1, 2.2 and 2.3, we have shown $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin \text{ORD}$ follows from (\boxplus) for our fixed n . Thus Claim 2 has been shown by induction. (Claim 2)

Consequently, if $\langle \cdot \rangle \notin \text{ORD}$ and $\overset{\text{I}}{y} = (a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ is a play according to s , then $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin \text{ORD}$ and in particular $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin D$ so that the play $\overset{\text{I}}{y}$ is not in the open set generated by D . (Case 2 and Theorem 2.12)

By Theorem 2.12, in any three-player game which satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, either player I has a strategy s_D to get into D or player II has a strategy s_D to stay out of D (depending on whether $\langle \rangle \in \text{ORD}_1^{3,D}$ or $\langle \rangle \notin \text{ORD}_1^{3,D}$). The strategy s_D naturally leads to a winning strategy for any three-player game in which $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, $O(D)$ is player I's payoff set, and in which player III has empty payoff.

Corollary 2.13. Every three-player game, which is open for player I, which satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ for some D that generates player I's payoff set, and in which player III has empty payoff, is determined. Moreover:

- (i) In any three-player game G in which player I has an open payoff set A Player I has a winning strategy for G exactly when $\langle \rangle \in \text{ORD}_1^{3,D}$ for some (any) D that generates A .⁸
- (ii) In any three-player game G in which player III has empty payoff, $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, and in which $O(D)$ is player I's payoff set, player II has a winning strategy for the game G exactly when $\langle \rangle \notin \text{ORD}_1^{3,D}$.

Proof. Let G be an open game which satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ for some D that generates A and in which player III has empty payoff. We show that either player I or player II has a winning strategy for the game G by considering the cases $\langle \rangle \in \text{ORD}_1^{3,D}$ and $\langle \rangle \notin \text{ORD}_1^{3,D}$.

⁸ If $\langle \rangle \in \text{ORD}_1^{3,D}$ (respectively $\langle \rangle \notin \text{ORD}_1^{3,D}$) for some D that generates A , then $\langle \rangle \in \text{ORD}_1^{3,D}$ (respectively $\langle \rangle \notin \text{ORD}_1^{3,D}$) for all D that generate A .

Case 1: $\langle \rangle \in \text{ORD}_I^{3,D}$.

In this case, we show that player I has a winning strategy for the game G by exactly the same proof presented in Case 1 of Corollary 2.7. By Theorem 2.12, player I has a strategy s to reach a position $\frac{1}{p}$ with ordinal zero. Let σ be the strategy in which player I plays according to s until he reaches such a $\frac{1}{p}$, and then he plays randomly. If $\frac{r}{y} = (y_0, y_1, y_2, \mathbf{K})$ is a play according to σ , then:

$$\exists n (y_0, y_1, \mathbf{K}, y_{n-1}, y_n) \text{ has ordinal zero.}$$

By the definition of $\text{ORD}_I^{3,D} 0$, $\exists i \leq n (y_0, y_1, \mathbf{K}, y_{i-1}, y_i) \in D$. Therefore since D generates the open set A , $\frac{1}{y} \in A$ and is a win for player I. Consequently, σ is a winning strategy for player I if $\langle \rangle \in \text{ORD}_I^{3,D}$.

Case 2: Suppose $\langle \rangle \notin \text{ORD}_I^{3,D}$.

In this case, we show that player II has a winning strategy for the game A . Since $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, by Theorem 2.12, player II has a strategy s to keep all positions according to s from being in D . Therefore, if $\frac{r}{y} = (y_0, y_1, y_2, \mathbf{K})$ is a play according to s , then $\forall n (y_0, y_1, \mathbf{K}, y_{n-1}, y_n) \notin D$: Therefore, since D generates the open set A , $\frac{1}{y} \notin A$ so that $\frac{1}{y}$ is not a win for player I. Since III has empty payoff $\frac{1}{y}$ is a win for player II.

Consequently, if $\langle \rangle \notin \text{ORD}_I^{3,D}$, then s is a winning strategy for player II.

Thus, either player I or player II has a winning strategy in the game A so that A is determined. , (Corollary 2.13)

The proofs of Theorem 2.12 and Corollary 2.13 easily generalize to give:

Corollary 2.14. Every κ -player game with players X and Y, in which player X has open payoff set $O(D)$ and in which $Z \xrightarrow[\text{D}]{\text{help}} X$ and player Z has empty payoff for any player Z different from X and Y, is determined. , (Corollary 2.14)

CHAPTER 3

DETERMINACY OF A THREE-PLAYER

BIASED GAME

In Chapter 3 we prove the main result of this thesis. In Section 2.2 we proved the determinacy of three-player biased open games with one open and one empty payoff set in which the player with the empty payoff set can't help the player with the open payoff set. In this section, we allow the player with the empty payoff set to have a nonempty payoff set, but to obtain determinacy two of the players will have open payoff sets and we shall add additional non-helping conditions stating that these two players cannot help each other.

Consider a three-player game with open payoff sets assigned to player I and player II. We know such games are not determined from Chapter 1 (see Theorem 1.1). However, with appropriate restrictions on the game tree and appropriate restrictions on the players "not helping" their opponents, we shall show games with these restrictions are determined. We introduced the concept of a player "not helping" his opponent in Section 2.2. In this section we expand upon our definition of "not helping" and upon the definition of ordinals of a position.

First let us note that the definition of ordinals of a position for player I in a three-player game as defined in Definition 2.8 (see Section 2.2) will not change for the three-player games considered in this section. The definition of ordinals of a position for

player X in a κ -player game¹ is similar to the definition of ordinals of a position for player I in Definition 2.8. At this time we generalize Definition 2.8 for a three player game to define $\text{ORD}_X^{\kappa,D}$, the ordinals of positions for a player X in a κ -player game with respect to a set D of positions.

Definition 3.1. Definition of ordinals of a position for κ -player games.

Let D be a collection of positions in a κ -player game. We assign certain positions an ordinal value for player X with respect to D as follows:

- (i) Let us have $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D} 0$ iff $\exists \overset{\Delta}{q} \in D \quad \overset{\Delta}{q} \subseteq \overset{\Gamma}{p}$,
- (ii) If the next move following position $\overset{\Delta}{p}$ belongs to player X, then $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D} \beta$ iff $\exists m \left[\overset{\Gamma}{p}(m) \text{ has } \text{ORD}_X^{\kappa,D} < \beta \right]$,
- (iii) If the next move following position $\overset{\Delta}{p}$ doesn't belong to player X, then $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D} \beta$ iff $\forall m \left[\overset{\Gamma}{p}(m) \text{ has } \text{ORD}_X^{\kappa,D} \leq \beta \right]$.

Let $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D} \beta$ iff $\overset{\Delta}{p}$ has ordinal β with respect to D and for player X in κ -player games. Let $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D}$ iff $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D} \beta$ for some ordinal β . Also let $\text{ORD}_X^{\kappa,D} \left(\overset{\Gamma}{p} \right) \downarrow$ (read: ordinal of $\overset{\Delta}{p}$ converges) iff $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D}$. Let $\text{ORD}_X^{\kappa,D} \left(\overset{\Delta}{p} \right) \uparrow$ (read: ordinal of $\overset{\Delta}{p}$ diverges) iff $\overset{\Gamma}{p} \notin \text{ORD}_X^{\kappa,D}$. , (Definition 3.1)

The following fact is clear from the definition of $\text{ORD}_X^{\kappa,D}$ for a κ -player game in which we assigned certain positions an ordinal value for player X with respect to a set D

¹ κ is an ordinal such that $\kappa \geq 3$.

of positions. This fact is a generalization of Fact 2.9 in Section 2.2 from a three-player game to a κ -player game.

Fact 3.2. If the $\text{ORD}_X^{\kappa,D}(a_0, a_1, \mathbf{K}, a_{m-1}, a_m) \downarrow$ and $\text{ORD}_X^{\kappa,D}(a_0, a_1, \mathbf{K}, a_{m-1}, a_m) = 0$, then any position that extends $(a_0, a_1, \mathbf{K}, a_{m-1}, a_m)$ also has $\text{ORD}_X^{\kappa,D} 0$. , (Fact 3.2)

By the same proof as that of Case 1 of Theorem 2.12, we have:

Lemma 3.3. Let D be a collection of positions. If $\langle \rangle \in \text{ORD}_X^{\kappa,D}$, then player X has a strategy to reach a position which has $\text{ORD}_X^{\kappa,D} 0$ in every κ -player game and therefore a position in D is reached when player X follows this strategy. More generally, if $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D}$, then player X has a strategy at $\overset{\Gamma}{p}$ to reach a position which has $\text{ORD}_X^{\kappa,D} 0$ in every κ -player game; therefore a position in D is reached if the position $\overset{\Gamma}{p} \in \text{ORD}_X^{\kappa,D}$ is reached and player X follows the above strategy.

Proof. (An outline of the proof). Let D be a collection of positions. As in Theorem 2.12, Case 1, player X plays moves that “strictly lower the ordinal value” of positions until a position with ordinal value zero is reached. By the definition of $\text{ORD}_X^{\kappa,D}$, any move by any player other than X cannot take a position in $\text{ORD}_X^{\kappa,D}$ to a position outside $\text{ORD}_X^{\kappa,D}$, or to a position with higher $\text{ORD}_X^{\kappa,D}$ -value. Consequently, since a decreasing sequence of ordinals must be finite, there is a position $\overset{\Gamma}{p}$ with an ordinal value of zero,

and by the definition of $\text{ORD}_X^{\kappa,D} 0$, $\exists p_0 \in D (p_0 \subseteq p)$. , (Lemma 3.3)

Just as we have Corollary 2.13-(i) to Theorem 2.12, we have the following by the analogous proof:

Corollary 3.4. Player X has a winning strategy in the κ -player game in which player X has open payoff and $\langle \rangle \in \text{ORD}_X^{\kappa,D}$ for some set D of positions which generates player X's (open) payoff set. , (Corollary 3.4)

In Section 2.2 we defined a player “not helping” player I with respect to D . This definition will apply to the three-player games in this section. We can generalize Definition 2.10 to define a player “not helping” player I in three-player games to a player “not helping” player X with respect to a set D of positions in κ -player games.

Definition 3.5. Definition of player Z not helping player X with respect to D , denoted by

$$Z \not\underset{D}{\text{help}} \rightarrow X .$$

A κ -player game G satisfies $Z \not\underset{D}{\text{help}} \rightarrow X$ (read: Z doesn't help X with respect to D)

iff X and Z are different players of the game G and the following holds:

If $p \in \text{ORD}_X^{\kappa,D}$ is a position such that the next move belongs to player Z and if $\exists z$ such that $p(z) \in \text{ORD}_X^{\kappa,D}$, then in the game G player Z may only play m such that $p(m) \in \text{ORD}_X^{\kappa,D}$. ,

Definition 3.6. A Sound Definition of player Z “not helping” player X with respect to D .

A κ -player game G satisfies $Z \not\underset{D}{\xrightarrow{\text{help}}} X$ iff X and Z are different players in the game G and the following holds:

If $\overset{r}{p} \notin \text{ORD}_X^{\kappa, D}$ is a position such that the next move belongs to player Z, then in $G \exists$ move z such that $\overset{r}{p}(z) \notin \text{ORD}_X^{\kappa, D}$ and player Z may only play such a move. $\quad ,$

If G satisfies $Z \not\underset{D}{\xrightarrow{\text{help}}} X$ and $\overset{r}{p} \notin \text{ORD}_X^{\kappa, D}$, then player Z must play a move z that keeps the position $\overset{r}{p}(z) \notin \text{ORD}_X^{\kappa, D}$ whenever such a z exists; we next show in Proposition 3.7 below that such a z always exists (in this case) so that such a G satisfies

$Z \not\underset{D}{\xrightarrow{\text{help}}} X$. Proposition 3.7 generalizes Proposition 2.11 from three-player games to κ -player games.

Proposition 3.7. Let D be a collection of positions. If $\overset{r}{p} \notin \text{ORD}_X^{\kappa, D}$ is a position such that the next move belongs to player Z (different from player X), then $\exists z$ such that

$\overset{r}{p}(z) \notin \text{ORD}_X^{\kappa, D}$. Hence the game G satisfies $Z \not\underset{D}{\xrightarrow{\text{help}}} X$ iff $Z \not\underset{D}{\xrightarrow{\text{help}}} X$.

Proof. Proof by contraposition. Let $\overset{r}{p}$ be a position such that the next move belongs to a player Z different than X. Suppose $\text{ORD}_X^{\kappa, D}(\overset{r}{p}(z)) \downarrow$ for any move z by player Z, i.e.

$\forall z \text{ ORD}_X^{K,D}(\overset{I}{p}(z)) \downarrow$. Let $\gamma = \sup_z [\text{ORD}_X^{K,D}(\overset{I}{p}(z))]$. By the definition of $\text{ORD}_X^{K,D}$ and since player Z is different from player X, $\text{ORD}_X^{K,D}(\overset{I}{p}) \downarrow$ and is γ . , (Proposition 3.7)

Recall that we are interested in the determinacy of three-player games G in which two players have open payoff, say players I and II respectively have open payoff sets $O(D)$ and $O(E)$ for some set of positions D and E . By Corollary 2.13-(i), if $\langle \rangle \in \text{ORD}_I^{3,D}$, then player I has a winning strategy for the game G . By the same proof, if $\langle \rangle \in \text{ORD}_{II}^{3,E}$, then player II has a winning strategy for the game G . Unfortunately, if $\langle \rangle \notin \text{ORD}_I^{3,D}$ and $\langle \rangle \notin \text{ORD}_{II}^{3,E}$, it doesn't follow that player III has a winning strategy for the game G .² However, we shall be able to prove the determinacy of G by considering a “restricted” version of $\text{ORD}_{II}^{3,E}$. Instead of the usual ordinals of a position for player II (in the game tree T), we will define for player II ordinals of positions in a tree T^{LD} , where

$$T^{LD} = \{(a_0, a_1, K, a_{n-2}, a_{n-1}) \mid \forall i < n (a_0, a_1, K, a_{i-1}, a_i) \notin \text{ORD}_I^{3,D}\}.$$

T^{LD} is “the tree of non-losing positions” for the players other than I. It is possible for T^{LD} to be the empty set, but we shall show that when $\langle \rangle \notin \text{ORD}_I^{3,D}$, T^{LD} is a game tree.

Definition 3.8. Definition of a game tree T .

A set T of positions is called a *game tree* iff

- (i) (closed downward) $\forall \overset{I}{p} \in T \forall \overset{I}{q} \subseteq \overset{I}{p} \overset{I}{q} \in T$,

² In the game G of Theorem 1.1, it is easy to verify $\langle \rangle \notin \text{ORD}_I^{3,D}$ and $\langle \rangle \notin \text{ORD}_{II}^{3,E}$ where $D = \emptyset$ and $E = \{(a)\}$; however player III doesn't have a winning strategy.

(ii) (extensions exist) $\forall \overset{\ulcorner}{p} \in T \exists m \overset{\ulcorner}{p}'(m) \in T$

Let T_q^{\ulcorner} be the set of all moves x such that $\overset{\ulcorner}{q}'(x) \in T$, i.e. $x \in T_q^{\ulcorner}$ iff $\overset{\ulcorner}{q}'(x) \in T$. ,

Definition 3.1 naturally generalizes to defining the ordinals of a position in game trees:

Definition 3.9. Definition of ordinals of a position for player Y in a game tree T for κ -player games.

Let T be a game tree, let E be a collection of positions, and let $\overset{\ulcorner}{p} \in T$. Inductively on γ , we define position $\overset{\ulcorner}{p} \in T$ having $\text{ORD}_{Y,E}^{\kappa,T} \gamma$:

- $\overset{\ulcorner}{p} \in \text{ORD}_{Y,E}^{\kappa,T} 0$ iff $\exists \overset{\ulcorner}{q} \in E (\overset{\ulcorner}{q} \subseteq \overset{\ulcorner}{p})$,
- If the next move following position $\overset{\ulcorner}{p}$ belongs to player Y, then $\overset{\ulcorner}{p} \in \text{ORD}_{Y,E}^{\kappa,T} \gamma$ iff $\exists m \in T_p^{\ulcorner} [\overset{\ulcorner}{p}'(m) \text{ has } \text{ORD}_{Y,E}^{\kappa,T} < \gamma]$,
- If the next move following position $\overset{\ulcorner}{p}$ doesn't belong to player Y, then $\overset{\ulcorner}{p} \in \text{ORD}_{Y,E}^{\kappa,T} \gamma$ iff $\forall m \in T_p^{\ulcorner} [\overset{\ulcorner}{p}'(m) \text{ has } \text{ORD}_{Y,E}^{\kappa,T} \leq \gamma]$.

Let $\overset{\ulcorner}{p} \in \text{ORD}_{Y,E}^{\kappa,T} \gamma$ iff $\overset{\ulcorner}{p}$ has ordinal γ with respect to E and for player Y in the game tree

T for κ -player games. Let $\overset{\ulcorner}{p} \in \text{ORD}_{Y,E}^{\kappa,T}$ iff $\overset{\ulcorner}{p} \in \text{ORD}_{Y,E}^{\kappa,T} \gamma$ for some ordinal γ . Also let

$\text{ORD}_{Y,E}^{\kappa,T}(\overset{\ulcorner}{p}) \downarrow$ (read: ordinal $\text{ORD}_{Y,E}^{\kappa,T}$ of $\overset{\ulcorner}{p}$ converges) iff $\overset{\ulcorner}{p} \in \text{ORD}_{Y,E}^{\kappa,T}$. Let $\text{ORD}_{Y,E}^{\kappa,T}(\overset{\ulcorner}{p}) \uparrow$

(read: ordinal $\text{ORD}_{Y,E}^{\kappa,T}$ of $\overset{\ulcorner}{p}$ diverges) iff $\overset{\ulcorner}{p} \notin \text{ORD}_{Y,E}^{\kappa,T}$. □

Notice $\text{ORD}_I^{3,D} = \text{ORD}_{I,D}^{3,T}$ if T is the tree of all possible positions. Besides $\text{ORD}_I^{3,D}$, in this section we are also interested in $\text{ORD}_{II,E}^{3,T}$, where E will generate player II's open payoff set in Theorem 3.1 (see below). From Definition 3.5, notice that $\text{ORD}_{II,E}^{3,T}$ is defined as follows:

Let T be a game tree, let E be a set of positions, and let $\overset{r}{q} \in T^{1,D}$. Inductively on γ , position $\overset{r}{q} \in T^{1,D}$ having $\text{ORD}_{II,E}^{3,T^{1,D}} \gamma$ is defined as follows:

- (i) $\overset{r}{q}$ has $\text{ORD}_{II,E}^{3,T^{1,D}} 0$ iff $\exists \overset{r}{p} \in E$ ($\overset{r}{p} \subseteq \overset{r}{q}$),
- (ii) $\overset{r}{q} = (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n})$ has $\text{ORD}_{II,E}^{3,T^{1,D}} \gamma$ iff

$$\exists f(3n+1) = a_{3n+1} \left[\overset{r}{q} \right] (f(3n+1)) \in T^{1,D} \text{ and } \text{ORD}_{II,E}^{3,T^{1,D}} \left[\overset{r}{q} \right] (f(3n+1)) < \gamma,$$
- (iii) $\overset{r}{q} = (a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1})$ has $\text{ORD}_{II,E}^{3,T^{1,D}} \gamma$ iff

$$\forall f(3n+2) = a_{3n+2} \left[\overset{r}{q} \right] (f(3n+2)) \in T^{1,D} \Rightarrow \text{ORD}_{II,E}^{3,T^{1,D}} \left[\overset{r}{q} \right] (f(3n+2)) \leq \gamma,$$
- (iv) $\overset{r}{q} = (a_0, a_1, \mathbf{K}, a_{3n+1}, a_{3n+2})$ has $\text{ORD}_{II,E}^{3,T^{1,D}} \gamma$ iff

$$\forall f(3n+3) = a_{3n+3} \left[\overset{r}{q} \right] (f(3n+3)) \in T^{1,D} \Rightarrow \text{ORD}_{II,E}^{3,T^{1,D}} \left[\overset{r}{q} \right] (f(3n+3)) \leq \gamma.$$

Definition 3.10. Let $\text{ORD}_{II,E}^{3,I,D} =_{\text{df}} \text{ORD}_{II,E}^{3,T}$, where $T = T^{1,D}$. Recall $\overset{r}{p} \in T^{1,D}$ iff

$$\forall \overset{r}{q} \subseteq \overset{r}{p} \text{ } \text{ORD}_I^{3,D}(\overset{r}{q}) \uparrow.$$

We need to define a player “not helping” player Y with respect to a set E of positions in a three-player game.

Definition 3.11: Definition of player Z not helping player Y in T with respect to E,

denoted by $Z \xrightarrow[\cancel{E}]{T, \text{help}} Y$.

A three-player game G satisfies $Z \xrightarrow[\cancel{E}]{T, \text{help}} Y$ (read: Z doesn't "help" player Y in T with respect to E) iff Y and Z are (different) players of the game G and the following holds:

If $q \in T \setminus \text{ORD}_{Y,E}^{3,T}$ and if $\exists z$ such that $q'(z) \in T \setminus \text{ORD}_{Y,E}^{3,T}$, then in the game G player Z may only play m such that $q'(m) \in T \setminus \text{ORD}_{Y,E}^{3,T}$.

Definition 3.12. A Sound Definition of player Z not "helping" player Y in T with respect to E.

A three-player game G satisfies $Z \xrightarrow[\cancel{E}]{T, \text{help}} Y$ iff Y and Z are different players in the game G and the following holds:

If $q \in T \setminus \text{ORD}_{Y,E}^{3,T}$ is a position such that the next move belongs to player Z, then in G \exists move m such that $q'(m) \in T \setminus \text{ORD}_{Y,E}^{3,T}$ and player Z may only play such a move.

Note that $Z \xrightarrow[\cancel{E}]{\text{help}} Y$ abbreviates $Z \xrightarrow[\cancel{E}]{T, \text{help}} Y$ if T is the game tree of all positions.

By the following proposition, G satisfies $Z \xrightarrow[\cancel{E}]{T, \text{help}} Y$ iff G satisfies

$Z \xrightarrow[\cancel{E}]{T, \text{help}} Y$.

Proposition 3.13. If $\overset{\mathbf{r}}{q} \in T \setminus \text{ORD}_{Y,E}^{3,T}$ is a position such that the next move belongs to player Z (different from player Y), then $\exists m$ such that $\overset{\mathbf{r}}{q'}(m) \in T \setminus \text{ORD}_{Y,E}^{3,T}$. Hence the game G satisfies $Z \xrightarrow[\text{E}]{\text{T, help}} Y$ iff G satisfies $Z \xrightarrow[\text{E}]{\text{T, help}} Y$.

Proof by contraposition: Let $\overset{\mathbf{1}}{p}$ be a position such that the next move belong so a player other than Y. Suppose $\text{ORD}_{Y,E}^{3,T}(\overset{\mathbf{r}}{q'}(z)) \downarrow$ for any move z by player Z, i.e.

$\forall z \in T_{\overset{\mathbf{r}}{q}} \overset{\mathbf{r}}{q'}(z) \in \text{ORD}_{Y,E}^{3,T}$. Let $\gamma = \sup_{z \in T_{\overset{\mathbf{r}}{q}}} [\text{ORD}_{Y,E}^{3,T}(\overset{\mathbf{r}}{q'}(z))]$. By the definition of $\text{ORD}_{Y,E}^{3,T}$,

$\text{ORD}_{Y,E}^{3,T}(\overset{\mathbf{r}}{q'}(z)) \downarrow$ and $\overset{\mathbf{1}}{q}$ has $\text{ORD}_{Y,E}^{3,T} \gamma$. , (Proposition 3.13)

Recall that in this section we shall be interested in three-player games G in which player I and II have open payoff sets and which satisfy some conditions which will make it possible for us to obtain determinacy (of such games). These conditions are:

III $\xrightarrow[\text{D}]{\text{help}} \text{I}$, II $\xrightarrow[\text{D}]{\text{help}} \text{I}$, and I $\xrightarrow[\text{E}]{\text{T, help}} \text{II}$, for some D and E that respectively generate

the open payoff sets for players I and II. Any such D and E will satisfy the following:

Definition of \perp : For sets D and E of positions, $D \perp E$ iff $\forall \overset{\mathbf{1}}{p} \in D \ \forall \overset{\mathbf{1}}{q} \in E \ \overset{\mathbf{1}}{p} \perp \overset{\mathbf{1}}{q}$,

where $\overset{\mathbf{1}}{p} \perp \overset{\mathbf{1}}{q}$ means $\overset{\mathbf{1}}{p}$ and $\overset{\mathbf{1}}{q}$ are incompatible, i.e. if $\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{i-1}, a_i)$ and

$\overset{\mathbf{r}}{q} = (b_0, b_1, \mathbf{K}, b_{j-1}, b_j)$, then $\exists k(k < i, k < j, \text{ and } a_i \neq b_j)$.³ ,

³ Recall $T^{\text{LD}} = \{(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \mid \forall i < n (a_0, a_1, \mathbf{K}, a_{i-1}, a_i)\}$.

Definition of P: For positions $\overset{\ulcorner}{p}$ and $\overset{\ulcorner}{q}$, $\overset{\ulcorner}{p} P \overset{\ulcorner}{q}$ iff $\overset{\ulcorner}{p} \not\perp \overset{\ulcorner}{q}$, i.e., $\overset{\ulcorner}{p}$ and $\overset{\ulcorner}{q}$ are compatible, meaning $\overset{\ulcorner}{p} \supseteq \overset{\ulcorner}{q}$ or $\overset{\ulcorner}{q} \supseteq \overset{\ulcorner}{p}$.

From Fact 2.9 we know that any move that extends a position that has $\text{ORD}_I^{3,D} 0$ also has an ordinal value of zero (with respect to $\text{ORD}_I^{3,D}$). We shall show that any position that extends a position with $\text{ORD}_{II,E}^{3,T^{LD}} 0$ will also have such ordinal value zero (with respect to $\text{ORD}_{II,E}^{3,T^{LD}}$). We do this in three steps:

Proposition 3.14.

- I. If $\text{ORD}_I^{3,D}(\overset{\ulcorner}{p}) \downarrow$, then $\exists \overset{\ulcorner}{q} \supseteq \overset{\ulcorner}{p}$ such that $\text{ORD}_I^{3,D}(\overset{\ulcorner}{q}) = 0$.
- II. If $\text{ORD}_{II,E}^{3,T^{LD}}(\overset{\ulcorner}{p}) = 0$ and $D \perp E$, then $\forall \overset{\ulcorner}{q} \supseteq \overset{\ulcorner}{p}$ $\text{ORD}_I^{3,D}(\overset{\ulcorner}{q}) \uparrow$.
- III. If $\text{ORD}_{II,E}^{3,T^{LD}}(\overset{\ulcorner}{p}) = 0$ and $D \perp E$, then

$$\forall \overset{\ulcorner}{q} \supseteq \overset{\ulcorner}{p} \left[\overset{\ulcorner}{q} \in T^{LD} \text{ and } \overset{\ulcorner}{q} \in \text{ORD}_{II,E}^{3,T^{LD}} \text{ and } \overset{\ulcorner}{q} \text{ has } \text{ORD}_{II,E}^{3,T^{LD}} 0 \right].$$

Proof of Part I. The proof is the same as that of Case 1 of Theorem 2.3. Let $\text{ORD}_I^{3,D}(\overset{\ulcorner}{p}) \downarrow$. Show that $\exists \overset{\ulcorner}{q}$ that extends $\overset{\ulcorner}{p}$ such that $\text{ORD}_I^{3,D}(\overset{\ulcorner}{q}) = 0$. Let $\overset{\ulcorner}{p}'$ be a position of least length that extends $\overset{\ulcorner}{p}$ and whose last move is made by player III⁴ ($\overset{\ulcorner}{p}'$ completes the round that $\overset{\ulcorner}{p}$ lies in). Then any move of $\overset{\ulcorner}{p}'$ not included in $\overset{\ulcorner}{p}$ is not made by player I (otherwise, $\overset{\ulcorner}{p}'$ and $\overset{\ulcorner}{p}$ would belong to different rounds). Hence by the

⁴ Possibly $\overset{\ulcorner}{p}' = \overset{\ulcorner}{p}(x)$.

definition of $\text{ORD}_1^{3,D}$, $\hat{p}' \in \text{ORD}_1^{3,D}$ since $\hat{p} \in \text{ORD}_1^{3,D}$. Set $(y_0, y_1, \mathbf{K}, y_{3\hat{l}-2}, y_{3\hat{l}-1}) = \hat{p}'$,

where $3\hat{l} = \ln(\hat{p}')$.⁵

We define a play $\hat{y} = (y_0, y_1, y_2, \mathbf{K})$ that extends \hat{p}' such that for all $i \geq \ln(\hat{p}')$:

$$(*)_i \quad (y_0, y_1, \mathbf{K}, y_{i-1}, y_i) \text{ has } \text{ORD}_1^{3,D},$$

$$(**)_i \quad \text{ORD}_1^{3,D}(y_0, y_1, \mathbf{K}, y_{i-1}, y_i) = 0 \text{ or}$$

$$\text{ORD}_1^{3,D}(y_0, y_1, \mathbf{K}, y_{i-1}, y_i) \mathbf{R}_i \text{ORD}_1^{3,D}(y_0, y_1, \mathbf{K}, y_{i-2}, y_{i-1})$$

$$\text{where } \mathbf{R}_i = \begin{cases} < & \text{if } i = 3k, \text{ for some } k \\ \leq & \text{if } i \neq 3k, \text{ for all } k \end{cases}.$$

To define such a \hat{y} , we *inductively* define each round of play extending \hat{p}' . Suppose we have $(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1})$ that extends \hat{p}' (possibly is equal to it) and,

$$(*)_i \text{ and } (**)_i \text{ hold for } \ln(\hat{p}') \leq i < 3l.$$

Next we notice that if $3l = \ln(\hat{p}')$, then $(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1}) = \hat{p}'$ and is $\in \text{ORD}_1^{3,D}$. If

$3l > \ln(\hat{p}')$, then $(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1}) \in \text{ORD}_1^{3,D}$ by $(*)_{3l-1}$. In either case,

$(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1}) \in \text{ORD}_1^{3,D}$. Hence either $\text{ORD}_1^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1}) > 0$ or

$\text{ORD}_1^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1}) = 0$.

We next define the moves $y_{3l}, y_{3l+1}, y_{3l+2}$ of the next round of play. The definition of y_{3l} is divided into two cases:

$$(i) \text{ORD}_1^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1}) > 0 \text{ or } (ii) \text{ORD}_1^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1}) = 0.$$

Definition of y_{3l} for case (i): $(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1})$ has $\text{ORD}_1^{3,D} > 0$.

⁵ We use \ln to note the length of the position.

Then in this case, by the definition of $\text{ORD}_I^{3,D}$, $\exists x$ such that $\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, x) < \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1})$. Since such an x exists, play I picks a y_{3l} such that $\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, y_{3l}) \downarrow$ and $\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, y_{3l}) < \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1})$. Then $(*)_{3l}$ and $(**)_{3l}$ hold.

Definition of y_{3l} for case (ii): $(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1})$ has $\text{ORD}_I^{3,D} = 0$.

In this case, by Fact 2.9, any move that extends $(y_0, y_1, \mathbf{K}, y_{3l-2}, y_{3l-1})$ will have $\text{ORD}_I^{3,D} > 0$. So player I may pick any y_{3l} . Again $(*)_{3l}$ and $(**)_{3l}$ hold.

Let players II and III respectively pick their favorite y_{3l+1} and y_{3l+2} . We show that $(*)_{3l+1}$, $(*)_{3l+2}$, $(**)_{3l+1}$, and $(**)_{3l+2}$ hold.

By the definition of $\text{ORD}_I^{3,D}$,

$$\forall x \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l}, x) \downarrow \text{ and}$$

$$\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l}, x) \leq \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, y_{3l}).$$

In particular, $\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l}, y_{3l+1}) \downarrow$ and is $\leq \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, y_{3l})$ for $x = y_{3l+1}$; therefore $(*)_{3l+1}$ and $(**)_{3l+1}$ hold.

By the definition of $\text{ORD}_I^{3,D}$,

$$\forall x \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l+1}, x) \downarrow \text{ and}$$

$$\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l+1}, x) \leq \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l}, y_{3l+1}).$$

In particular $\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l+1}, y_{3l+2}) \downarrow$ and is $\leq \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l}, y_{3l+1})$ for $x = y_{3l+2}$; therefore $(*)_{3l+2}$ and $(**)_{3l+2}$ hold.

We define the play of each round to continue in the above manner. Defining the moves $y_{3l}, y_{3l+1}, y_{3l+2}$ of players I, II, and III, respectively, gives an inductive definition of each round of play extending $\frac{1}{p'}$.

By the definition of the play $\overset{\mathbf{r}}{y} = (y_0, y_1, y_2, \mathbf{K})$, $\overset{\mathbf{r}}{y} \supseteq \overset{\mathbf{r}}{p'}$, $(*)_i$ and $(**)_i$ hold for $i \geq \ln(\frac{\mathbf{r}}{p'})$. Since we have for $3l \geq \ln(\frac{\mathbf{r}}{p'})$, $\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l+n-1}, y_{3l+n}) \downarrow$ for $n = 0, 1, 2, 3, \mathbf{K}$,

$$\text{ORD}_I^D(y_0, y_1, \mathbf{K}, y_{3l+2}, y_{3l+3}) = 0, \text{ or}$$

$$\begin{aligned} \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l+2}, y_{3l+3}) &< \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l+1}, y_{3l+2}) \\ &\leq \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l}, y_{3l+1}) \leq \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, y_{3l}). \end{aligned}$$

Therefore, if $3l \geq \ln(\frac{\mathbf{r}}{p'})$ and $\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, y_{3l}) \neq 0$ then

$$\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, y_{3l}) > \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l+2}, y_{3l+3}). \text{ If } \forall \hat{l} \text{ such that } 3l \geq \ln(\frac{\mathbf{r}}{p'}),$$

$$\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3l-1}, y_{3l}) \neq 0, \text{ then}$$

$$\begin{aligned} \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3\hat{l}-1}, y_{3\hat{l}}) &> \text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3\hat{l}+2}, y_{3\hat{l}+3}) \\ &> \text{ORD}_I^{3,D}(y_0, y_1, y_2, \mathbf{L}, y_{3\hat{l}+5}, y_{3\hat{l}+6}) > \text{ORD}_I^{3,D}(y_0, y_1, y_2, \mathbf{L}, y_{3\hat{l}+8}, y_{3\hat{l}+9}) > \mathbf{L} \end{aligned}$$

which gives an infinite decreasing sequence of ordinals. Since a decreasing sequence of

ordinals must be finite, $\exists \hat{l} \geq \hat{l} \text{ ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3\hat{l}-1}, y_{3\hat{l}}) \downarrow$ and

$$\text{ORD}_I^{3,D}(y_0, y_1, \mathbf{K}, y_{3\hat{l}-1}, y_{3\hat{l}}) = 0.$$

Let $\overset{\mathbf{r}}{q} =_{\text{df}} (y_0, y_1, \mathbf{K}, y_{3\hat{l}-1}, y_{3\hat{l}})$; then $\text{ORD}_I^{3,D}(\overset{\mathbf{r}}{q}) = 0$, $\overset{\mathbf{r}}{q} \supseteq \overset{\mathbf{r}}{p'} \supseteq \overset{\mathbf{r}}{p}$, and the conclusion of (I) has been shown. , (Part I)

Proof of Part II. We assume that $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0$, and that the conclusion of II is false:

$\exists \overset{\text{I}}{q} \supseteq \overset{\text{I}}{p}$ $\text{ORD}_I^{3,\text{D}}(\overset{\text{I}}{q}) \downarrow$. We show the hypothesis of II is false, i.e. we show $D \not\subseteq E$. Since $\text{ORD}_I^{3,\text{D}}(\overset{\text{I}}{q}) \downarrow$, by I, $\exists \overset{\text{I}}{q'} \supseteq \overset{\text{I}}{q}$ such that $\text{ORD}_I^{3,\text{D}}(\overset{\text{I}}{q'}) = 0$. By the definition of $\text{ORD}_I^{3,\text{D}} 0$, $\exists \overset{\text{I}}{q_0} \in D$ such that $\overset{\text{I}}{q_0} \subseteq \overset{\text{I}}{q'}$. By the definition of $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}} 0$, and since $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0$, $\exists \overset{\text{I}}{p_0} \in E$ such that $\overset{\text{I}}{p_0} \subseteq \overset{\text{I}}{p}$. But $\overset{\text{I}}{p_0} \subseteq \overset{\text{I}}{p} \subseteq \overset{\text{I}}{q} \subseteq \overset{\text{I}}{q'}$. Thus $\overset{\text{I}}{q'}$ extends $\overset{\text{I}}{q_0} \in D$ and $\overset{\text{I}}{p_0} \in E$ so that $D \not\subseteq E$. Consequently part (II) has been shown. , (Part II)

Proof of Part III. Assume $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0$ and pick $\overset{\text{I}}{q} \supseteq \overset{\text{I}}{p}$. We shall show $\overset{\text{I}}{q} \in \text{T}^{\text{LD}}$,

$\overset{\text{I}}{q} \in \text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$, and $\overset{\text{I}}{q}$ has $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}} 0$. By II, $\text{ORD}_I^{3,\text{D}}(\overset{\text{I}}{q}) \uparrow$ so that $\overset{\text{I}}{q} \in \text{T}^{\text{LD}}$. Since

$\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0$, by the definition of such ordinal value zero, $\exists \overset{\text{I}}{p_0} \in E$ ($\overset{\text{I}}{p_0} \subseteq \overset{\text{I}}{p}$).

$\overset{\text{I}}{p_0} \subseteq \overset{\text{I}}{p} \subseteq \overset{\text{I}}{q}$. Thus, $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{q}) = 0$ (and $\overset{\text{I}}{q} \in \text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$) since $\overset{\text{I}}{q} \supseteq \overset{\text{I}}{p_0}$, $\overset{\text{I}}{p_0} \in E$, and

$\overset{\text{I}}{q} \in \text{T}^{\text{LD}}$. Consequently part III has been shown. , (Part III and Proposition 3.14)

In Proposition 3.14 we have shown that any move that extends a position with $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$ value of zero will also have such an ordinal value of zero. In Corollary 3.16 below we show that player II has a winning strategy for *appropriate* three-player games in which player I and II have open payoff sets $O(D)$ and $O(E)$ respectively and in which $\langle \rangle \in \text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$. Corollary 3.16 follows from the following:

Lemma 3.15. If $\langle \rangle \in \text{ORD}_{\text{II},E}^{3,T^{1,D}}$, $D \perp E$, and G is a three-player game that satisfies

$\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, then player II has a strategy to reach a position in G with $\text{ORD}_{\text{II},E}^{3,T^{1,D}} = 0$, and

therefore a position in E is reached when player II follows this strategy.

Proof. Let D and E be sets of positions in a three-player game G which satisfies

$\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$. Assume $\langle \rangle \in \text{ORD}_{\text{II},E}^{3,T^{1,D}}$. We construct a strategy for player II in which he

plays “to strictly lower the ordinal values of positions” in $\text{ORD}_{\text{II},E}^{3,T^{1,D}}$ until a position with

ordinal value zero is reached. More generally, if $\overset{\mathbf{r}}{q} \in \text{ORD}_{\text{II},E}^{3,T^{1,D}}$, then player II has a

strategy at $\overset{\mathbf{r}}{q}$ to reach a position which has $\text{ORD}_{\text{II},E}^{3,T^{1,D}} = 0$ in appropriate three-player games;

therefore a position in E is reached if the position $\overset{\mathbf{r}}{q} \in \text{ORD}_{\text{II},E}^{3,T^{1,D}}$ is reached and player II

follows the above strategy. Define a strategy for player II as follows:

$$\text{Let } s(a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) = \left\{ \begin{array}{l}
\mu a_{3n+1} \text{ such that} \\
(a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \in \mathbb{T}^{\text{LD}} \\
\text{and } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \downarrow \\
\text{and } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) \downarrow \\
\text{and } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) \\
< \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) \\
\text{if } \exists x \text{ such that } (a_0, a_1, \mathbf{K}, a_{3n}, x) \in \mathbb{T}^{\text{LD}} \\
\text{and } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n}, x) \downarrow \\
\text{and } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) \downarrow \\
\text{and } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n}, x) \\
< \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}), \\
\\
\text{\& otherwise, i.e.} \\
\forall x \in \mathbb{T}^{\text{LD}} \left[\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) \uparrow \right. \\
\text{or } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n}, x) \uparrow, \text{ or} \\
\left. \left\{ \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) \downarrow \text{ and} \right. \right. \\
\left. \left. \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n}, x) \downarrow \text{ and} \right. \right. \\
\left. \left. \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n-1}, a_{3n}) \leq \right. \right. \\
\left. \left. \left. \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3n}, x) \right\} \right].
\end{array} \right.$$

Let $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ be a legal play according to s . We shall show

$$\exists n \left[\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow \text{ and } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) = 0 \right].$$

Claim. $\forall n \left[\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow \right]$ and either $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) = 0$

or $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \mathbf{R}_n \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1})$ where

$$\mathbf{R}_n = \left\{ \begin{array}{l} < \text{ if } n = 3l + 1, \text{ for some } l \\ \leq \text{ if } n \neq 3l + 1, \text{ for all } l \end{array} \right\}.$$

Considering the cases “ $n = 3l$ ”, “ $n = 3l + 1$ ”, and “ $n = 3l + 2$ ” we show by induction that

the Claim holds. Fix $n \in \omega$ and let $\overset{\mathbf{r}}{q} = (a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1})$. We first note that

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \downarrow. \#$$

This follows by the Induction Hypothesis when $n - 1 \in \mathbb{Y}$ and otherwise by the

assumption that $\langle \rangle \in \text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}}$. Let $\gamma = \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, a_2, \mathbf{L}, a_{n-2}, a_{n-1})$.

Case 1: $n = 3k$ for some $k \geq 0$.

Show that $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \downarrow$ and either $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) = 0$

$$\text{or } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \leq \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}).$$

Recall $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \downarrow$ and $\gamma = \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1})$.

By the definition of $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}$,

$$\forall x \in \mathbf{T}_{\overset{\mathbf{r}}{q}}^{\text{LD}} \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k-1}, x) \downarrow \quad \boxed{1}$$

and is $\leq \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) = \gamma$ regardless of whether $\gamma = 0$ or $\gamma > 0$. Since

by # $(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \in \mathbf{T}^{\text{LD}}$, $(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \notin \text{ORD}_{\text{I}}^{3,\text{D}}$; then by the

definition of $\text{ORD}_{\text{I}}^{3,\text{D}}$, $\forall z (a_0, a_1, \mathbf{K}, a_{3k-1}, z) \notin \text{ORD}_{\text{I}}^{3,\text{D}}$. In particular

$(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \notin \text{ORD}_1^{3,D}$ and so $a_{3k} \in T_q^{\text{LD}}$ Hence by $\square 1$

$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \downarrow$ and is $\leq \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1})$.

Case 2: $n = 3k + 1$ for some $k \geq 0$.

Show that $\text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \downarrow$ and either

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) = 0 \text{ or}$$

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) < \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}).$$

Recall $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k}) \downarrow$ and $\gamma = \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k})$.

If $\gamma = 0$, then by Proposition 3.9 $\forall x (a_0, a_1, \mathbf{K}, a_{3k}, x)$ has $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} 0$. If $\gamma > 0$, then by

the definition of $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}$, $\exists x \in T_q^{\text{LD}} \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, x) \downarrow$ and

$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, x) < \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) = \gamma$. By the definition of s,

$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \downarrow$ and is $< \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k})$ since

$(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1})$ is according to s.

Case 3 $n = 3k + 2$ for some $k \geq 0$.

By a similar argument as that presented in Case 1⁶, we

show that $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) \downarrow$ and either

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) = 0 \text{ or}$$

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) \leq \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}).$$

⁶ To stay out of $\text{ORD}_1^{3,D}$, we use $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ in this case which is III's turn, whereas we used the definition of $\text{ORD}_1^{3,D}$ in case (i) where it is player I's turn.

Recall $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \downarrow$ and $\gamma = \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1})$. By the definition of $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}$,

$$\forall x \in \text{T}_q^{1,\text{D}} \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k+1}, x) \downarrow. \quad \boxed{2}$$

Since $(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \in \text{T}^{\text{LD}}$, $(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \notin \text{ORD}_I^{3,\text{D}}$, and since

$\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, player III must play a move x such that $(a_0, a_1, \mathbf{K}, a_{3k+1}, x) \notin \text{ORD}_I^{3,\text{D}}$. In

particular $(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) \notin \text{ORD}_I^{3,\text{D}}$ so that $a_{3k+2} \in \text{T}_q^{1,\text{D}}$. Hence by $\boxed{2}$,

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k+2}, a_{3k+2}) \downarrow \text{ and is } \leq \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}).$$

Consequently by cases (i), (ii), and (iii), we have shown the claim by induction.

, (Claim)

By the claim, $\forall n \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \downarrow$ and if the

$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \neq 0$, then

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) < \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \leq$$

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \leq \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-3}, a_{3k-2}).$$

Hence if $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \neq 0$, then

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-3}, a_{3k-2}) < \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-6}, a_{3k-5}) <$$

$$< \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k-9}, a_{3k-8}) < L < \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1).$$

Therefore, if $\forall k \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \neq 0$, then

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1) > \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0, a_1, a_2, a_3, a_4)$$

$$> \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k+3}, a_{3k+4}) > \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k+6}, a_{3k+7}) > L$$

which gives an infinite decreasing sequence of ordinals. Since a decreasing sequence of ordinals must be finite, $\exists k \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} (a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) = 0$. By the definition of

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} 0, \exists i \leq k (a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \in E. \quad , \text{ (Lemma 3.15)}$$

We now show that player II's strategy s_E in Lemma 3.15 above naturally leads to a winning strategy for player II in certain three-player games:

Corollary 3.16. If G is a three player game in which:

- (i) players I and II respectively have open payoff sets $O(D)$ and $O(E)$,
- (ii) $\langle \rangle \in \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}$, and
- (iii) G satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$,

then player II has a winning strategy for the game G .

Proof. Let G be a three-player game as described in the hypothesis. Let D and E generate player I and II's open payoff sets, respectively. By Lemma 3.2 player II has a strategy s to get to a position $\frac{1}{p}$ of $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} 0$. Let σ be the strategy in which player II plays according to s until he reaches such a $\frac{1}{p}$ and then he plays randomly. Let

$\overset{\text{I}}{y} = (y_0, y_1, y_2, \mathbf{K})$ be a play according σ . Then there is a position

$(y_0, y_1, \mathbf{K}, y_{n-1}, y_n) \in \text{T}^{\text{LD}}$ with $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} 0$. By the definition of $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} 0$,

$\exists i \leq n$ $(y_0, y_1, \mathbf{K}, y_{i-1}, y_i) \in E$ so that $\frac{1}{y} \in O(E)$. Hence $\frac{1}{y}$ is a winning play for player II and σ is a winning strategy. , (Corollary 3.16)

We have shown that either player I or II has a winning strategy for certain three-player games when $\langle \rangle$ is either in $\text{ORD}_I^{3,D}$ or $\text{ORD}_{II,E}^{3,T^{LD}}$ for appropriate D and E . We next show that player III will have a winning strategy for appropriate three-player games when $\langle \rangle$ is neither in $\text{ORD}_I^{3,D}$ nor in $\text{ORD}_{II,E}^{3,T^{LD}}$.

Lemma 3.17. If G is a three-player game in which:

- (i) G satisfies $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\text{E}]{T^{LD} \text{ help}} \text{II}$, and
- (ii) $\langle \rangle \notin \text{ORD}_I^{3,D}$ and $\langle \rangle \notin \text{ORD}_{II,E}^{3,T^{LD}}$,

then player III has a strategy to keep all positions out of both $\text{ORD}_I^{3,D}$ and $\text{ORD}_{II,E}^{3,T^{LD}}$ in the game G and in particular from being in D or E . Hence any play according to this strategy is neither in $O(D)$ nor $O(E)$.

Proof. Let $\langle \rangle \notin \text{ORD}_I^{3,D}$ and $\langle \rangle \notin \text{ORD}_{II,E}^{3,T^{LD}}$ and show player III has a strategy such that all positions according to this strategy are neither in $\text{ORD}_I^{3,D}$ nor in $\text{ORD}_{II,E}^{3,T^{LD}}$. Therefore all positions are in $T^{LD} \setminus \text{ORD}_{II,E}^{3,T^{LD}}$. Define a strategy for player III as follows:

$$\text{Let } s(a_0, a_1, \mathbf{K}, a_{3n}, a_{3n+1}) = \begin{cases} \mu a_{3n+2} \\ (a_0, a_1, \mathbf{K}, a_{3n+1}, a_{3n+2}) \in T^{\text{I,D}} \setminus \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}} \\ \text{if } T^{\text{I,D}} \setminus \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}} \neq \emptyset, \\ 8 \text{ otherwise, if there is no such} \\ a_{3n+2}. \end{cases}$$

We shall show that if $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ is a legal play according to s , then

$\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \in T^{\text{I,D}} \setminus \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}}$, i.e. that no move by player I nor player II can take a position in $T^{\text{I,D}} \setminus \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}}$ to a position outside $T^{\text{I,D}} \setminus \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}}$.

Claim. $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \in T^{\text{I,D}} \setminus \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}}$ if $(a_0, a_1, a_2, \mathbf{K})$ is a legal play according to s .

Fix a play $(a_0, a_1, a_2, \mathbf{K})$ according to s . Considering the cases “ $n = 3l$ ”, “ $n = 3l + 1$ ”, and “ $n = 3l + 2$ ”, we show by induction that the Claim holds. Fix $n \in \omega$. Note that:

$$(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \in T^{\text{I,D}} \setminus \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}}, \text{ i.e. } (\mathfrak{a}\mathfrak{a})$$

$$\text{ORD}_I^{3,\text{D}}(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \uparrow \text{ and } \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}}(a_0, a_1, \mathbf{K}, a_{n-2}, a_{n-1}) \uparrow,$$

as this follows by the Induction Hypothesis when $n - 1 \in \mathbb{Y}$ and otherwise, when $n = 0$, by assumption (ii) in the hypothesis to this lemma that $\langle \rangle \notin \text{ORD}_I^{3,\text{D}}$ and $\langle \rangle \notin \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{I,D}}}$.

Case 1: $n = 3k$ for some $k \geq 0$ (i.e. it is player I's turn).

By $(\mathfrak{a}\mathfrak{a})$ $(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \notin \text{ORD}_I^{3,\text{D}}$, so that by the definition of $\text{ORD}_I^{3,\text{D}}$,

$$\forall x (a_0, a_1, \mathbf{K}, a_{3k-1}, x) \notin \text{ORD}_I^{3,\text{D}}.$$

In particular $(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \notin \text{ORD}_I^{3,D}$.

By $(\alpha\alpha)$ $(a_0, a_1, \mathbf{K}, a_{3k-2}, a_{3k-1}) \notin \text{ORD}_{II,E}^{3,T^{LD}}$ so that, since G satisfies $I \xrightarrow[\text{E}]{T^{LD} \text{ help}} II$,

player I must play x such that $(a_0, a_1, \mathbf{K}, a_{3k-1}, x) \notin \text{ORD}_{II,E}^{3,T^{LD}}$. In particular

$$(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \notin \text{ORD}_{II,E}^{3,T^{LD}}.$$

We have shown that

$$(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \notin \text{ORD}_I^{3,D} \text{ and } (a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \notin \text{ORD}_{II,E}^{3,T^{LD}}.$$

Therefore we have $(a_0, a_1, a_2, \mathbf{L}, a_{3k-1}, a_{3k}) \in T^{LD} \setminus \text{ORD}_{II,E}^{3,T^{LD}}$.

Case 2: $n = 3k + 1$ for some $k \geq 0$ (i.e. it is player II's turn).

By $(\alpha\alpha)$ $(a_0, a_1, a_2, \mathbf{L}, a_{3k-1}, a_{3k}) \notin \text{ORD}_I^{3,D}$ so that since G satisfies $II \xrightarrow[\text{D}]{\text{help}} I$,

player II must play an x such that $(a_0, a_1, \mathbf{K}, a_{3k-1}, x) \notin \text{ORD}_I^{3,D}$. In particular

$$(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \notin \text{ORD}_I^{3,D}.$$

By $(\alpha\alpha)$ $(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \notin \text{ORD}_{II,E}^{3,T^{LD}}$ so that by the definition of $\text{ORD}_{II,E}^{3,T^{LD}}$,

$$\forall x (a_0, a_1, \mathbf{K}, a_{3k}, x) \notin \text{ORD}_{II,E}^{3,T^{LD}}.$$

In particular, $(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \notin \text{ORD}_{II,E}^{3,T^{LD}}$.

We have shown that

$$(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \notin \text{ORD}_I^{3,D} \text{ and } (a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \notin \text{ORD}_{II,E}^{3,T^{LD}}.$$

Therefore $(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) \in T^{LD} \setminus \text{ORD}_{II,E}^{3,T^{LD}}$.

Case 3: $n = 3k + 2$ for some $k \geq 0$, (i.e. it is player III's turn).

In this case, we use player III's strategy to stay out of $\text{ORD}_I^{3,D}$ and $\text{ORD}_{II,E}^{3,T^{LD}}$. Let

$\overset{\Gamma}{p} = (a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1})$. We first note that by the definition of $\text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}$, we have:

$$\text{if } \forall x \in \text{T}_p^{\text{LD}} \text{ORD}_{\text{II}}^{3, \text{T}^{\text{LD}}}(\overset{\Gamma}{p})(x) \downarrow, \text{ then } \text{ORD}_{\text{II}}^{3, \text{T}^{\text{LD}}}(\overset{\Gamma}{p}) \downarrow. \quad (7)$$

Recall $\text{ORD}_{\text{II}}^{3, \text{T}^{\text{LD}}}(\overset{\Gamma}{p}) \uparrow$ by (2); therefore we have by (7) that

$$\exists x \in \text{T}_p^{\text{LD}}(\overset{\Gamma}{p})(x) \notin \text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}.$$

Hence $\exists x \overset{\Gamma}{p}(x) \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}$ so that, by the definition of s and since

$(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2})$ is according to s ,

$$(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}.$$

By Cases 1, 2, and 3, we have shown $(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}$ for our fixed n . Therefore, by induction, we have proved our claim that

$$\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}. \text{ In particular for every } n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n)$$

doesn't have $\text{ORD}_1^{3, \text{D}}$ -value zero so $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin D$. Also by the claim, for

every $n, (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \in \text{T}^{\text{LD}}$ and doesn't have $\text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}$ -value zero; so

$$\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin E. \quad , \text{ (Claim)}$$

Consequently if $\langle \rangle \notin \text{ORD}_1^{3, \text{D}}, \langle \rangle \notin \text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}$ and $\overset{\Gamma}{y} = (a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ is a play

according to s , then $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}$ (by the claim) and in

particular $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin D$ and $\forall n (a_0, a_1, \mathbf{K}, a_{n-1}, a_n) \notin E$. Therefore, the

⁷ If $\forall x \in \text{T}_p^{\text{LD}}(a_0, a_1, a_2, \mathbf{K}, a_{3k+1}, x) \in \text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}$, then by the definition of, $\text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}$,

$$\text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}(a_0, a_1, a_2, \mathbf{K}, a_{3k}, a_{3k+1}) \downarrow \text{ and equals } \sup_{x \in \text{T}_p^{\text{LD}}} \left[\text{ORD}_{\text{II}, \text{E}}^{3, \text{T}^{\text{LD}}}(a_0, a_1, a_2, \mathbf{K}, a_{3k+1}, x) \right].$$

play $\overset{1}{y}$ is neither in the open set generated by D nor in the open set generated by E .

, (Lemma 3.17)

We now show that the strategy s used by player III above is a winning strategy for certain three-player games:

Corollary 3.18. Player III has a winning strategy for every three-player game in which: player I and II respectively have open payoff sets,

(i) G satisfies $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}}, \text{help}} \text{II}$ for some D and E that

respectively generate player I and player II's payoff sets,

(ii) $\langle \rangle \notin \text{ORD}_I^{3,D}$ and $\langle \rangle \notin \text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$, i.e. $\langle \rangle \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$.

Proof. Let G be a three-player game which satisfies the hypothesis above. By Lemma 3.3 player III has a strategy s , such that all positions according to s are neither in D nor E . Let $\overset{\text{r}}{y} = (y_0, y_2, y_3, \mathbf{K})$ be a play according to s . Then:

$$\forall n (y_0, y_1, \mathbf{K}, y_{n-1}, y_n) \notin D \text{ and } \forall n (y_0, y_1, \mathbf{K}, y_{n-1}, y_n) \notin E.$$

Hence

$$\overset{1}{y} \notin O(D) \text{ and } \overset{1}{y} \notin O(E).$$

Therefore, $\overset{1}{y}$ is neither a win for player I nor II. Thus $\overset{1}{y}$ is a win for player III.

Consequently we have shown that s is a winning strategy for player III.

, (Corollary 3.18)

Corollaries 3.1, 3.2 and 3.18 together imply that certain three-player games are determined.

Theorem 3.19. Determined are three-player games G which satisfy the following:

- (i) players I and II have open payoff sets,
- (ii) $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$, and $\text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}}, \text{help}} \text{II}$ for some D and E that respectively generate player I and player II's payoff sets.

Moreover, for any such game G ,

- a. Player I has a winning strategy for G when $\langle \rangle \in \text{ORD}_I^{3,D}$.
- b. Player II has a winning strategy for G when $\langle \rangle \in \text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$.
- c. Player III has a winning strategy for G when $\langle \rangle \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$.

Proof: Let G , D , and E be as in the hypothesis to the theorem so that (i) and (ii) hold.

(a) follows from Corollary 3.4, (b) follows from Corollary 3.2, and (c) follows from

Corollary 3.18. G is determined by (a), (b), and (c). (Theorem 3.19)

Just as we consolidated Corollaries 3.4, 3.16, and 3.3 into Theorem 3.20, we can consolidate Lemmas 3.3, 3.15, and 3.17 to obtain:

Proposition 3.20. If $D \perp E$ and G is a three player game which satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$,

$\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$, and $\text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}}, \text{help}} \text{II}$, then:

- (i) when $\langle \rangle \in \text{ORD}_I^{3,D}$, player I has a strategy to reach a position in D ,
- (ii) when $\langle \rangle \in \text{ORD}_{II,E}^{3,T^{LD}}$, player II has a strategy to reach a position in E , and
- (iii) when $\langle \rangle \in T^{LD} \setminus \text{ORD}_{II,E}^{3,T^{LD}}$, player III has a strategy to keep all positions out of both D and E .

, (Proposition 3.20)

In the next chapter we show that determinacy in Theorem 3.19 fails if we drop any one of the three non-helping conditions from the hypothesis (ii) of Theorem 3.19. In Chapter 5, we verify that Theorem 3.19 holds even if we permute the roles of the players. In Chapter 6, we generalize the result of Chapters 3 and 5 from three-player biased games to $(n+1)$ -player biased games.

The non-helping conditions restrict players from intentionally helping an opponent on any move, but a nice result from Theorem 3.19 restricts a player from making a move which results in a different ordinal value than before the move was made.

Corollary 3.21. Determined is any infinite three-player game of perfect information in which:

- (1) at most one player has a payoff set that is not open,
- (2) at every position, there is a move m such that at the resulting position, no player other than possibly the player making the move m has a winning strategy, and
- (3) each is required to make such a move m . , (Corollary 3.21)

OLD CHAPTER 3 MATERIAL

DO NOT COPY

If $\text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0$, then $\forall x \left[\overset{\text{I}}{p}(x) \in \text{T}^{\text{LD}} \text{ and } \overset{\text{I}}{p}(x) \in \text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}} \text{ and } \overset{\text{I}}{p}(x) \text{ has } \text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}} 0 \right]$.

Proof: Let $\text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0$. Pick an arbitrary x . We will show:

$$\overset{\text{I}}{p}(x) \in \text{T}^{\text{LD}}, \overset{\text{I}}{p}(x) \in \text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}} \text{ and } \overset{\text{I}}{p}(x) \text{ has } \text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}} 0.$$

Since $\text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0$, by the definition of $\text{ORD}_{\text{II,E}}^{\text{T}^{\text{LD}}}$ there exists $\overset{\text{I}}{q}_j \subseteq \overset{\text{I}}{p}$ and $\overset{\text{I}}{p} \in \text{T}^{\text{LD}}$.

First we will show $\overset{\text{I}}{p}(x) \in \text{T}^{\text{LD}}$ by contradiction.

Assume $\overset{\text{I}}{p}(x) \notin \text{T}^{\text{LD}}$; therefore $\overset{\text{I}}{p}(x)$ has ORD_I^{D} . Let $\overset{\text{I}}{p}'$ be a position of least length that extends $\overset{\text{I}}{p}(x)$ and whose last move is made by player III (possibly $\overset{\text{I}}{p}' = \overset{\text{I}}{p}(x)$).

Then any move of $\overset{\text{I}}{p}'$ not included in $\overset{\text{I}}{p}(x)$ is not made by player I. Hence, by the definition of ORD_I^{D} , $\overset{\text{I}}{p}' \in \text{ORD}_I^{\text{D}}$ since $\overset{\text{I}}{p}(x) \in \text{ORD}_I^{\text{D}}$. Set $(y_0, y_1, y_2, \text{L}, y_{3\hat{l}-2}, y_{3\hat{l}-1}) = \overset{\text{I}}{p}'$,

where $3\hat{l} = \ln(\overset{\text{I}}{p}')$.⁸

We define a play $\overset{\text{I}}{y} = (y_0, y_1, y_2, \text{L})$ that extends $\overset{\text{I}}{p}'$ such that for all $i \geq \ln(\overset{\text{I}}{p}')$:

$$(*)_i \quad (y_0, y_1, y_2, \text{L}, y_{i-1}, y_i) \text{ has } \text{ORD}_I^{\text{D}},$$

$$(**)_i \quad \text{ORD}_I^{\text{D}}(y_0, y_1, y_2, \text{L}, y_{i-1}, y_i) = 0,$$

or

$$\text{ORD}_I^{\text{D}}(y_0, y_1, y_2, \text{L}, y_{i-1}, y_i) \text{ R}_i \text{ORD}_I^{\text{D}}(y_0, y_1, y_2, \text{L}, y_{i-2}, y_{i-1})$$

⁸ We use \ln to note the length of the position.

$$\text{where } R_i = \begin{cases} < & \text{if } i = 3k, \text{ for some } k \\ \leq & \text{if } i \neq 3k, \text{ for all } k \end{cases} .$$

Suppose we have $(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1})$ that extends $\frac{1}{p'}$ (possibly is equal to it) and,

$(*)_i$ and $(**)_i$ hold for $\ln(\frac{r'}{p'}) \leq i < 3l$. Next we notice that if $3l = \ln(\frac{r'}{p'})$, then

$(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1}) = \frac{r'}{p'}$ and is $\in \text{ORD}_1^D$. If $3l > \ln(\frac{r'}{p'})$, then

$(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1}) \in \text{ORD}_1^D$ by $(*)_{3l-1}$. In either case,

$(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1}) \in \text{ORD}_1$. Hence either $\text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1}) > 0$ or

$\text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1}) = 0$.

We next define the moves $y_{3l}, y_{3l+1}, y_{3l+2}$ of the next round of play. The definition of y_{3l} is divided into two cases:

i) $\text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1}) > 0$

and

ii) $\text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1}) = 0$.

i) $(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1})$ has $\text{ORD}_1^D > 0$; then by the definition of ORD_1^D , $\exists x$ such that $\text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-1}, x) < \text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1})$. Since such an x

exists, pick y_{3l} such that $\text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-1}, y_{3l}) \downarrow$ and

$\text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-1}, y_{3l}) < \text{ORD}_1^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1})$. Then $(*)_{3l}$ and $(**)_{3l}$

hold.

ii) $(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1})$ has $\text{ORD}_I^D = 0$; then by Fact 2.2, any move that extends $(y_0, y_1, y_2, \mathbf{L}, y_{3l-2}, y_{3l-1})$ will have $\text{ORD}_I^D > 0$. So player I will pick any y_{3l} . Again $(*)_{3l}$ and $(**)_{3l}$ hold.

Let player II pick his favorite y_{3l+1} and player III pick his favorite y_{3l+2} . By the definition of ORD_I^D , $\forall x \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l}, x) \downarrow$ and $\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l}, x) \leq \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-1}, y_{3l})$. In particular these hold for $x = y_{3l+1}$; therefore $(*)_{3l+1}$ and $(**)_{3l+1}$ hold.

By the definition of ORD_I^D , $\forall x \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l+1}, x) \downarrow$ and $\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l+1}, x) \leq \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l}, y_{3l+1})$. In particular these hold for $x = y_{3l+2}$; therefore $(*)_{3l+2}$ and $(**)_{3l+2}$ hold.

We define the play of each round to continue in the above manner. Defining the moves $y_{3l}, y_{3l+1}, y_{3l+2}$, of players I, II, and III respectively, gives an inductive definition of each round of play extending $\frac{1}{p}$.

By the definition of the play $\overset{\mathbf{r}}{y} = (y_0, y_1, y_2, \mathbf{L})$, $\overset{\mathbf{r}}{y} \supseteq \overset{\mathbf{r}}{p}$, $(*)_i$ and $(**)_{i-1}$ hold for $i \geq \ln(\frac{\mathbf{r}}{p})$. In particular we have for $3l \geq \ln(\frac{\mathbf{r}}{p})$, $\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l+n}) \downarrow$ for $n = 0, 1, 2, 3$; and $\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l+2}, y_{3l+3}) = 0$, or $\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l+2}, y_{3l+3}) < \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l+1}, y_{3l+2}) \leq \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l}, y_{3l+1}) \leq \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-1}, y_{3l})$. Therefore, if $3l \geq \ln(\frac{\mathbf{r}}{p})$ and $\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-1}, y_{3l}) \neq 0$ then

$\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-1}, y_{3l}) > \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l+2}, y_{3l+3})$. If

$\forall l$ such that $3l \geq \ln(\overset{I}{p}')$, $\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3l-1}, y_{3l}) \neq 0$, then

$$\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3\hat{i}}) > \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3\hat{i}+3}) >$$

$$\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3\hat{i}+6}) > \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3\hat{i}+9}) >$$

$\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3\hat{i}+12}) > \mathbf{L}$. Since there is no infinite decreasing sequence of

ordinals, $\exists \ell \text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3\ell_0}, y_{3\ell_0}) \downarrow$ and $\text{ORD}_I^D(y_0, y_1, y_2, \mathbf{L}, y_{3\ell_0}, y_{3\ell_0}) = 0$.

By definition $(y_0, y_1, y_2, \mathbf{L}, y_{3\ell_0}, y_{3\ell_0})$ extends some $\overset{I}{p}_j$ so that $\overset{I}{y}$ is a win for player I.

$(y_0, y_1, y_2, \mathbf{L})$ extends $\overset{I}{p}'$ which in turn extends $\overset{I}{q}_i$ so that $\overset{I}{y}$ is a win for player II. We

have a contradiction. Therefore $\overset{I}{p}'(x) \in T^{\text{LD}}$. Therefore, since $\overset{I}{q}_i \subseteq \overset{I}{p} \subseteq \overset{I}{p}'(x)$,

$\overset{I}{p}'(x) \in \text{ORD}_{\text{II,E}}^{\text{LD}}$ and $\text{ORD}_{\text{II,D}}^{\text{LE}}(\overset{I}{p}'(x)) = 0$, by the definition of $\text{ORD}_{\text{II,E}}^{\text{LD}}$. PROPOSITION 3.? \square

Let A and B be disjoint open sets. We will define a three-player game

$G = G_{\text{restricted}}(A, B, \overline{A} \cup B)$ which is played along a game tree T depending on A and B ,

and in which A and B will be the payoff sets for player I and player II respectively. Since

A is open, there exist finite sequences $\overset{I}{p}_1, \overset{I}{p}_2, \overset{I}{p}_3, \mathbf{L}$ such that:

$$f \in A \text{ iff } \exists i \overset{I}{p}_i \subseteq f.$$

Since B is open, there exist finite sequences $\overset{I}{q}_1, \overset{I}{q}_2, \overset{I}{q}_3, \mathbf{L}$ such that:

$$h \in B \text{ iff } \exists j \overset{I}{q}_j \subseteq h.$$

Consider the game $G = G_{\text{restricted}}(A, B, \overline{A} \cup B)$:

$$\begin{array}{ccccccccccc}
& & \text{I} & a_0 & & a_3 & & & a_{3n} & & \\
\text{G:} & \text{II} & & a_1 & & a_4 & & \text{L} & & a_{3n+1} & & \text{L} & . \\
& \text{III} & & & a_2 & & a_5 & & & & a_{3n+2} & &
\end{array}$$

We now state the desired restrictions for each player. These restrictions are defined in terms of ORD_I and $\text{ORD}_{II}^{\text{I}}$. We intend that the stated restrictions are to be followed whenever possible by the designated player.

Rule 1: Player II will not “help” player I whenever possible. More precisely we mean if $\overset{\text{I}}{p} = (a_0, a_1, a_2, \text{L}, a_{3k-1}, a_{3k}) \notin \text{ORD}_I$ and $\exists x$ such that

$(a_0, a_1, a_2, \text{L}, a_{3k}, x) = \overset{\text{I}}{p}(x) \notin \text{ORD}_I$, then player II must play a_{3k+1} such that

$(a_0, a_1, a_2, \text{L}, a_{3k}, a_{3k+1}) \notin \text{ORD}_I$.

Rule 2: Whenever possible player III will not “help” player I. More precisely we mean if $\overset{\text{I}}{p} = (a_0, a_1, a_2, \text{L}, a_{3k}, a_{3k+1}) \notin \text{ORD}_I$ and $\exists x$ such that

$(a_0, a_1, a_2, \text{L}, a_{3k+1}, x) = \overset{\text{I}}{p}(x) \notin \text{ORD}_I$, then player III must play a_{3k+2} such that

$(a_0, a_1, a_2, \text{L}, a_{3k+1}, a_{3k+2}) \notin \text{ORD}_I$.

Rule 3: Player I will not “help” player II whenever possible. More precisely we mean if $\overset{\text{I}}{q} = (a_0, a_1, a_2, \text{L}, a_{3k-2}, a_{3k-1}) \in \text{T}^I \setminus \text{ORD}_{II}^{\text{I}}$ and $\exists x$ such that

$(a_0, a_1, a_2, \text{L}, a_{3k-1}, x) = \overset{\text{I}}{q}(x) \in \text{T}^I \setminus \text{ORD}_{II}^{\text{I}}$, then player I must play a_{3k} such that

$(a_0, a_1, a_2, \text{L}, a_{3k-1}, a_{3k}) \in \text{T}^I \setminus \text{ORD}_{II}^{\text{I}}$.

At this time, by describing what moves are allowable, we have implicitly described the game tree along which the game is played. Notice that the above rules require that a certain player does not “help” another player “whenever possible”. Basically the above rules are of the form: If a position is out of the ordinals, then the non-helping player must

play so that the extended position is also out of the ordinals whenever such a move exists.

We now show that such a move always exists whenever the original position doesn't have an ordinal. This will also show that "whenever possible" can be dropped from the above rules.

Proposition p1. If $(a_0, a_1, a_2, L, a_{3k-1}, z) \notin \text{ORD}_I$, then $\forall z (a_0, a_1, a_2, L, a_{3k-1}, z) \notin \text{ORD}_I$.

Proof by contraposition: Assume $\exists z (a_0, a_1, a_2, L, a_{3k-1}, z) \in \text{ORD}_I$ and show

$$(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) \in \text{ORD}_I.$$

Since $(a_0, a_1, a_2, L, a_{3k-1}, z) \in \text{ORD}_I$, then $\text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, z) \downarrow$. Let

$\gamma = \text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, z)$. By the definition of ORD_I ,

$$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) \downarrow \text{ and is } \geq \gamma + 1.$$

END PROPOSITION P1 \square

Proposition R1: If $\overset{r}{p} = (a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \notin \text{ORD}_I$, then

$$\exists x (a_0, a_1, a_2, L, a_{3k}, x) \notin \text{ORD}_I.$$

Proof by contraposition: Suppose $\forall x (a_0, a_1, a_2, L, a_{3k}, x) \in \text{ORD}_I$, then

$$\forall x \text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, x) \downarrow.$$

Let $\gamma = \sup_x [\text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, x)]$. By the definition of ORD_I ,

$$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \downarrow \text{ and is } \leq \gamma.$$

END PROPOSITION R1 \square

Therefore if $(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \notin \text{ORD}_I$ then there is an $x = a_{3k+1}$ such that

$(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \notin \text{ORD}_I$. Since an a_{3k+1} exists in this case, Rule 1 can be

replaced by Rule $\hat{1}$: If $\overset{\text{I}}{p} = (a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \notin \text{ORD}_I$ then player II must play

a_{3k+1} such that $(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \notin \text{ORD}_I$.

Proposition R2: If $\overset{\text{I}}{p} = (a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \notin \text{ORD}_I$, then

$\exists x(a_0, a_1, a_2, L, a_{3k+1}, x) \notin \text{ORD}_I$.

Proof by contraposition: Suppose $\forall x(a_0, a_1, a_2, L, a_{3k+1}, x) \in \text{ORD}_I$, then

$\forall x \text{ORD}_I(a_0, a_1, a_2, L, a_{3k+1}, x) \downarrow$.

Let $\alpha = \sup_x [\text{ORD}_I(a_0, a_1, a_2, L, a_{3k+1}, x)]$. By the definition of ORD_I ,

$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \downarrow$ and is $\leq \alpha$.

END PROPOSITION R2 \square

Therefore if $(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \notin \text{ORD}_I$ then there is an $x = a_{3k+2}$ such that $(a_0, a_1, a_2, L, a_{3k+1}, a_{3k+2}) \notin \text{ORD}_I$. Since an a_{3k+2} exists in this case, Rule 2 can be replaced by Rule $\hat{2}$: If $\overset{\text{I}}{p} = (a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \notin \text{ORD}_I$ then player III must play a_{3k+2} such that $(a_0, a_1, a_2, L, a_{3k+1}, a_{3k+2}) \notin \text{ORD}_I$.

Therefore if $(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) \in \text{T}^I \setminus \text{ORD}_{\text{II}}^{\text{T}^I}$, then player I must play $a_{3k} \in \text{T}^I \setminus \text{ORD}_{\text{II}}^{\text{T}^I}$ such that $(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \in \text{T}^I \setminus \text{ORD}_{\text{II}}^{\text{T}^I}$. Since an a_{3k} exists in this case, Rule 3 can be replaced by Rule $\hat{3}$: If

$\overset{\text{I}}{q} = (a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) \in \text{T}^I \setminus \text{ORD}_{\text{II}}^{\text{T}^I}$, then player I must play $a_{3k} \in \text{T}^I$ such that

$(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \in \text{T}^I \setminus \text{ORD}_{\text{II}}^{\text{T}^I}$.

Proposition R3: If $\overset{I}{q} = (a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) \in T^I \setminus \text{ORD}_{II}^I$, then $\exists x$ such that $(a_0, a_1, a_2, L, a_{3k-1}, x) \in T^I \setminus \text{ORD}_{II}^I$.

Proof by contraposition: Suppose $\forall x \in T_{(\bullet)}^I (a_0, a_1, a_2, L, a_{3k-1}, x) \in \text{ORD}_{II}^I$, then $\forall x \in T_{(\bullet)}^I \left[\text{ORD}_{II}^I (a_0, a_1, a_2, L, a_{3k-1}, x) \downarrow \right]$.

Let $\gamma = \sup_{x \in T_{(\bullet)}^I} \left[\text{ORD}_{II}^I (a_0, a_1, a_2, L, a_{3k-1}, x) \right]$. By the definition of ORD_{II}^I ,

$\text{ORD}_{II}^I (a_0, a_1, a_2, L, a_{3k-1}) \downarrow$ and is $\leq \gamma$.

END PROPOSITION R3 \square

THEREOM 3. If $\langle \rangle \in \text{ORD}_I$, player I has a strategy to get a position which has $\text{ORD}_I 0$. If $\langle \rangle \notin \text{ORD}_I$, then player II or player III has a strategy to get to a winning position.

Case I: Let $\langle \rangle \in \text{ORD}_I$.

Proposition I-B: If $\langle \rangle \in \text{ORD}_I$, then player I has a strategy to get to a position which has $\text{ORD}_I 0$. This strategy is a winning strategy for player I. Let us first note that the second sentence of Proposition I-B follows from the first (in the case of $\langle \rangle \in \text{ORD}_I$).

Suppose player I has a strategy σ to get to a position of $\text{ORD}_I 0$. Let $\overset{I}{y}$ be a play according to σ , as described in Proposition I-B, then there is a position

$(y_0, y_1, y_2, L, y_{n-1}, y_n)$ with $\text{ORD}_I 0$. By the definition of $\text{ORD}_I 0$ and Fact 1, any play extending $(y_0, y_1, y_2, L, y_{n-1}, y_n)$ is a win for player I. In particular $\overset{I}{y}$ is a winning play for player I. Consequently σ is a winning strategy for player I.

Now let us prove if $\langle \rangle \in \text{ORD}_I$, then player I has a strategy to get to a position which has $\text{ORD}_I 0$.

Define a strategy for player I as follows:

$$\text{Let } s(a_0, a_1, a_2, L, a_{3n-2}, a_{3n-1}) = \begin{cases} \mu a_{3n} \text{ if } \exists x = a_{3n} \left[\text{ORD}_I(a_0, a_1, a_2, L, a_{3n-1}, x) \downarrow \right. \\ \text{and } \text{ORD}_I(a_0, a_1, a_2, L, a_{3n-2}, a_{3n-1}) \downarrow \\ \text{and } \text{ORD}_I(a_0, a_1, a_2, L, a_{3n-1}, x) < \\ \left. \text{ORD}_I(a_0, a_1, a_2, L, a_{3n-2}, a_{3n-1}) \right], \\ \\ \text{8 otherwise, i.e.} \\ \forall x \left[\text{ORD}_I(a_0, a_1, a_2, L, a_{3n-2}, a_{3n-1}) \uparrow \right. \\ \text{or } \text{ORD}_I(a_0, a_1, a_2, L, a_{3n-1}, x) \uparrow \\ \text{or } \left\{ \text{ORD}_I(a_0, a_1, a_2, L, a_{3n-2}, a_{3n-1}) \downarrow \right. \\ \text{and } \text{ORD}_I(a_0, a_1, a_2, L, a_{3n-1}, x) \downarrow \\ \text{and } \text{ORD}_I(a_0, a_1, a_2, L, a_{3n-2}, a_{3n-1}) \leq \\ \left. \left. \text{ORD}_I(a_0, a_1, a_2, L, a_{3n-1}, x) \right\} \right]. \end{cases}$$

Let $(a_0, a_1, a_2, L, a_{n-1}, a_n, L)$ be a legal, infinite play according to s . We will show

$$\exists n \left[\text{ORD}_I(a_0, a_1, a_2, L, a_{n-1}, a_n) \downarrow \text{ and } \text{ORD}_I(a_0, a_1, a_2, L, a_{n-1}, a_n) = 0 \right].$$

Claim I: $\forall n \left[\text{ORD}_I(a_0, a_1, a_2, L, a_{n-1}, a_n) \downarrow \text{ and either}$

$$\text{ORD}_I(a_0, a_1, a_2, L, a_{n-1}, a_n) = 0 \text{ or}$$

$$\text{ORD}_I(a_0, a_1, a_2, L, a_{n-1}, a_n) R_n \text{ORD}_I(a_0, a_1, a_2, L, a_{n-2}, a_{n-1}) \text{ where}$$

$$R_n = \begin{cases} < \text{ if } n = 3l, \text{ for some } l \\ \leq \text{ if } n \neq 3l, \text{ for all } l \end{cases}.$$

Case (i): Let $n = 3k$.

Assume the hypothesis is true for $\hat{n} < 3k$, then

$\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) \downarrow$. Show that $\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \downarrow$ and

either $\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) = 0$ or

$\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) < \text{ORD}_1(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1})$.

The $\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) \downarrow$ so let

$\gamma = \text{ORD}_1(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1})$. If $\gamma = 0$, then by Fact 1 (stated before Theorem),

$\forall x (a_0, a_1, a_2, L, a_{3k-1}, x)$ is also 0. In particular $\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \downarrow$

and is 0. If $\gamma > 0$, then by the definition of ORD_1 ,

$\exists x \text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, x) \downarrow$ and

$\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, x) < \text{ORD}_1(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) = \gamma$. Since

$(a_0, a_1, a_2, L, a_{3k-1}, a_{3k})$ is according to s, by the definition of s

$\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \downarrow$ and is $< \text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k-1})$.

Case (ii): Let $n = 3k + 1$.

Assume the hypothesis is true for $\hat{n} < 3k + 1$, then

$\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \downarrow$. Show that $\text{ORD}_1(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \downarrow$ and

either the $\text{ORD}_1(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) = 0$ or

$\text{ORD}_1(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \leq \text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k})$.

The $\text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \downarrow$ so let $\gamma = \text{ORD}_1(a_0, a_1, a_2, L, a_{3k-1}, a_{3k})$.

By the definition of ORD_1 , $\forall x \text{ORD}_1(a_0, a_1, a_2, L, a_{3k}, x) \downarrow$ and is

$\leq \text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) = \gamma$, regardless of whether $\gamma = 0$ or $\gamma > 0$. In particular

$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \downarrow$ and is $\leq \text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k})$.

Case (iii): Let $n = 3k+2$.

Assume the hypothesis is true for $\hat{n} < 3k+2$; then

$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \downarrow$. Show that $\text{ORD}_I(a_0, a_1, a_2, L, a_{3k+1}, a_{3k+2}) \downarrow$ and

either the $\text{ORD}_I(a_0, a_1, a_2, L, a_{3k+1}, a_{3k+2}) = 0$ or

$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k+1}, a_{3k+2}) \leq \text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, a_{3k+1})$.

The $\text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \downarrow$ so let $\gamma = \text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, a_{3k+1})$.

By the definition of ORD_I , $\forall x \text{ORD}_I(a_0, a_1, a_2, L, a_{3k+1}, x) \downarrow$ and is

$\leq \text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) = \gamma$ regardless of whether $\gamma = 0$ or $\gamma > 0$. In particular

$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k+1}, a_{3k+2}) \downarrow$ and is $\leq \text{ORD}_I(a_0, a_1, a_2, L, a_{3k}, a_{3k+1})$.

END CLAIM I \square

So by cases (i), (ii), and (iii) we have $\forall n \text{ORD}_I(a_0, a_1, a_2, L, a_{n-1}, a_n) \downarrow$. By the claim, if $\text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \neq 0$, then

$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) < \text{ORD}_I(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1})$

$\leq \text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k-2}) \leq \text{ORD}_I(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-3})$. Hence if the

$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \neq 0$, then $\text{ORD}_I(a_0, a_1, a_2, L, a_{3k-2}, a_{3k-3}) \neq 0$, and

$\text{ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) < \text{ORD}_I(a_0, a_1, a_2, L, a_{3k-4}, a_{3k-3})$

$< \text{ORD}_I(a_0, a_1, a_2, L, a_{3k-7}, a_{3k-6}) < L < \text{ORD}(a_0, a_1, a_2, a_3) < \text{ORD}_I(a_0)$. Therefore if

$\forall k \text{ ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) \neq 0$, then

$\text{ORD}_I(a_0) > \text{ORD}_I(a_0, a_1, a_2, a_3) > \text{ORD}_I(a_0, L, a_3, L, a_6) >$

$\text{ORD}_I(a_0, L, a_3, L, a_6, L, a_9) > L > \text{ORD}_I(a_0, L, a_3, L, a_{3k}, L, a_{3k+3}) > L$ which

gives an infinite decreasing sequence of ordinals. Since any decreasing sequence of

ordinals is finite, $\exists k \text{ ORD}_I(a_0, a_1, a_2, L, a_{3k-1}, a_{3k}) = 0$.

□

END PROPOSITION I - B

Case II: Let $\langle \rangle \notin \text{ORD}_I$ and $\langle \rangle \in \text{ORD}_{II}^{T^1}$.

Proposition II-C: If $\langle \rangle \notin \text{ORD}_I$ and $\langle \rangle \in \text{ORD}_{II}^{T^1}$, then player II has a winning strategy to get to a position which has $\text{ORD}_{II}^{T^1} 0$. This strategy is a winning strategy for player II.

Let us first note that the second sentence of Proposition II-C follows from the first (in the case of $\langle \rangle \notin \text{ORD}_I$ and $\langle \rangle \in \text{ORD}_{II}^{T^1}$). Suppose player II has a strategy θ to get to a position of $\text{ORD}_{II}^{T^1} 0$. Let \dot{y} be a play according to θ as described in Proposition II-C.

Then there is a position $(y_0, y_1, y_2, L, y_{n-1}, y_n) \in T^1$ with $\text{ORD}_{II}^{T^1} 0$. By the definition of $\text{ORD}_{II}^{T^1} 0$ any play extending $(y_0, y_1, y_2, L, y_{n-1}, y_n)$ is a win for player II. In particular \dot{y} is a winning play for player II. Consequently θ is a winning strategy for player II.

Now let us prove if $\langle \rangle \notin \text{ORD}_I$ and $\langle \rangle \in \text{ORD}_{II}^{T^1}$, then player II has a strategy to get to a position which has $\text{ORD}_{II}^{T^1} 0$.

Case III: Let $\langle \rangle \notin \text{ORD}_I$ and $\langle \rangle \notin \text{ORD}_{II}^{T^1}$.

Proposition III-D: If $\langle \cdot \rangle \notin \text{ORD}_I$ and $\langle \cdot \rangle \notin \text{ORD}_{II}^{T^I}$, then player III has a strategy such that all positions $\overset{I}{p}$ according to this strategy are not in ORD_I ($\overset{I}{p} \notin \text{ORD}_I$) and are not in $\text{ORD}_{II}^{T^I}$ ($\overset{I}{p} \notin T^I$ or $\overset{I}{p} \in T^I \setminus \text{ORD}_{II}^{T^I}$). This strategy is a winning strategy for player III. Let us note that the second sentence of Proposition III-D follows from the first (in the case of $\langle \cdot \rangle \notin \text{ORD}_I$ and $\langle \cdot \rangle \notin \text{ORD}_{II}^{T^I}$). Suppose player III has a strategy τ such that all positions according to τ are not in ORD_I and are not in $\text{ORD}_{II}^{T^I}$. Let $\overset{I}{y}$ be a play according to τ as described in Proposition III-D; then

(*) $\forall n (y_0, y_1, y_2, L, y_{n-1}, y_n) \notin \text{ORD}_I$ and (**) $\forall n (y_0, y_1, y_2, L, y_{n-1}, y_n) \notin \text{ORD}_{II}^{T^I}$. If $\overset{I}{y}$ were a win for player I, $\exists i \text{ORD}_I(y_0, y_1, y_2, L, y_{i-1}, y_i) = 0$. Hence by (*), $\overset{I}{y}$ is not a win for player I. If $\overset{I}{y}$ were a win for player II, $\exists j \overset{I}{q}_j \subseteq \overset{I}{y}$. Then $\overset{I}{q}_j = (y_0, y_1, y_2, L, y_{\ln(\overset{I}{q}_j)-1})$ and we have $\overset{I}{q}_j \in T^I$ by (*) so $(y_0, y_1, y_2, L, y_{\ln(\overset{I}{q}_j)-1})$ has $\text{ORD}_{II}^{T^I} 0$. Hence by (**), $\overset{I}{y}$ is not a win for player II. Thus $\overset{I}{y}$ is a win for player III. Consequently we have shown τ is a winning strategy for player III.

Now let us prove if $\langle \cdot \rangle \notin \text{ORD}_I$ and $\langle \cdot \rangle \notin \text{ORD}_{II}^{T^I}$, then player III has a strategy such that all positions according to this strategy are not in ORD_I and not in $\text{ORD}_{II}^{T^I}$. Define a strategy for player III as follows:

□

END PROPOSITION III-D

By propositions I, II, and III we have shown this a determined game. It is important to note our proof is dependent upon our using $ORD_{II}^{T^I}$ instead of ORD_{II} . If we try to use ORD_{II} (this notation has no restricted T^I) instead, our proof breaks down in trying to show that player III has a winning strategy. If we replace $ORD_{II}^{T^I}$ with ORD_{II} in Proposition III-D, the proof breaks down in case (iii) and the two cases we considered in case (iii) become (1) $\exists x(a_0, a_1, a_2, L, a_{3k+1}, x) \notin ORD_{II}$ and (2) $\forall x(a_0, a_1, a_2, L, a_{3k+1}, x) \in ORD_{II}$. As before case (2) doesn't occur. However, even though $\exists x(a_0, a_1, a_2, L, a_{3k+1}, x) \notin ORD_{II}$ and $(a_0, a_1, a_2, L, a_{3k+1}, a_{3k+2}) \notin ORD_I$ we aren't able to conclude that $\exists y(a_0, a_1, a_2, L, a_{3k+1}, y) \notin ORD_I$ and $\exists y(a_0, a_1, a_2, L, a_{3k+1}, y) \notin ORD_{II}$. However, we need such a y in order to avoid the probability of for each z , either player I or player II has a winning strategy at $(a_0, a_1, a_2, L, a_{3k+1}, z)$ - player I for some z 's and player II for the remaining z 's.

Remarks: We remark here about where the non-helping rules were used. When $\langle \rangle \in ORD_I$, the definition of ORD_I was used to define a winning strategy for player I and none of the non-helping rules were used. When $\langle \rangle \in ORD_{II}^{T^I}$, we used $III \xrightarrow{\text{help}} I$ to define a winning strategy for player II. When $\langle \rangle \notin ORD_I$ and $\langle \rangle \notin ORD_{II}^{T^I}$, we used $I \xrightarrow{\text{help}} II$ and $II \xrightarrow{\text{help}} I$ to define a winning strategy for player III.

CHAPTER 4

SHOWING THE DETERMINACY OF THE BIASED GAME (CHAPTER 3) IS OPTIMAL

In Chapter 3 we proved certain three-player biased games are determined (see Theorem 3.19). In this chapter, we show that such determinacy is optimal. We show that for any two of the three non-helping conditions, $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$, or $\text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}} \text{help}} \text{II}$, the collection of games which satisfy those two conditions is not determined. For each of the three conditions, we provide a nondetermined game in which that condition fails but the other two are satisfied.

In our first case we shall define a nondetermined three-player game G in which players I and II have open payoff sets and which satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}} \text{help}} \text{II}$. By Theorem 3.19, G cannot satisfy $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$ since otherwise G would be determined.

Theorem 4.1. Theorem 3.19 with $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$ omitted is false, i.e. there exists a nondetermined game G in which the payoff sets of players I and II are respectively open sets $O(D)$ and $\emptyset = O(\emptyset)$ and which satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\emptyset]{\text{T}^{\text{LD}} \text{help}} \text{II}$.

Proof. Let A be any set with at least two elements and let $M \in A$. We shall present a three-player (nondetermined) game $G = G_M^A(\mathcal{O}(D), \emptyset, \vec{\mathcal{O}}(D))$ that satisfies the above theorem:

$$G: \begin{array}{ccccccc} & \text{I} & a_0 & & a_3 & & a_{3n} \\ & \text{II} & & a_1 & & a_4 & \text{L} & & a_{3n+1} & & \text{L} & . \\ & \text{III} & & & a_2 & & & a_5 & & & & a_{3n+2} \end{array}$$

Player I wins iff $a_1 = M$. Player III wins iff $a_1 \neq M$. Player II never wins, and player II's payoff set is the open set $\mathcal{O}(\emptyset)$, which is just the empty set. Below we define D and in Claims 4.3, 4.4 and 4.5, we respectively prove that G satisfies $\text{III} \xrightarrow[\mathcal{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\emptyset]{\Gamma^{\text{LD}} \text{ help}} \text{II}$ and that G is not determined.

Let D be the collection of all finite sequences of the form (a_0, M) where $a_0 \in A$.

Player I's payoff set is the open set $\mathcal{O}(D)$. Note that:

$$f \in \mathcal{O}(D) \text{ iff } \exists a_0 (a_0, M) \subseteq f .$$

$$\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) = 0 \text{ iff } \overset{\Gamma}{p} \supseteq (a_0, M) \text{ for some } a_0 .$$

Let $a_1(\overset{\Gamma}{p})$ denote a_1 when $\overset{\Gamma}{p} = (a_0, a_1, a_2, \text{L} , a_{n-1}, a_n)$. Note that

$$\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) = 0 \text{ iff } \ln(\overset{\Gamma}{p}) \geq 2 \text{ and } a_1(\overset{\Gamma}{p}) = M .$$

Lemma 4.2. If $\ln(\overset{\Gamma}{p}) \leq 1$, then $\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) \uparrow$, i.e. $\overset{\Gamma}{p}$ has no $\text{ORD}_1^{3,D}$ -value.

Assume $\ln(\overset{\Gamma}{p}) \leq 1$. Suppose $\overset{\Gamma}{p}$ has $\text{ORD}_1^{3,D}$. First we note that $\exists \overset{\Gamma}{q} \supseteq \overset{\Gamma}{p}$ such that $\ln(\overset{\Gamma}{q}) = 1$ and $\overset{\Gamma}{q}$ has $\text{ORD}_1^{3,D}$.

If $\ln(\overset{\Gamma}{p}) = 1$, let $\overset{\Gamma}{q} = \overset{\Gamma}{p}$.

If $\ln(\overset{\mathbf{r}}{p}) = 0$, then $\overset{\mathbf{1}}{p} = \langle \ \ \rangle$. Recall $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{p}) \downarrow$. By the definition of $\text{ORD}_1^{3,D}0$, $\beta =_{\text{df}} \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{p}) > 0$ since $\ln(\overset{\mathbf{r}}{p}) < 1$. By the definition of $\text{ORD}_1^{3,D}$, $\exists a_0$ such that $\overset{\mathbf{r}}{p}(a_0) = (a_0)$ has $\text{ORD}_1^{3,D} < \beta$; in this case let $\overset{\mathbf{r}}{q} = \overset{\mathbf{r}}{p}(a_0) = (a_0)$.

So in either case, we have $\overset{\mathbf{1}}{q} \supseteq \overset{\mathbf{1}}{p}$, $\ln(\overset{\mathbf{r}}{q}) = 1$, and $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}) \downarrow$. Let $\delta = \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q})$.

By the definition of $\text{ORD}_1^{3,D}0$, $\delta > 0$. By the definition of $\text{ORD}_1^{3,D}$,

$$\begin{aligned} \forall a_1 \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1)) \downarrow \text{ and } \leq \delta, \\ \forall a_1 \forall a_2 \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1, a_2)) \downarrow \text{ and } \leq \delta, \\ \text{and } \forall a_1 \forall a_2 \exists a_3 \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1, a_2, a_3)) \downarrow \text{ and } < \delta. \end{aligned}$$

Pick $a_1 \neq M$, pick any a_2 , and select an a_3 that satisfies $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1, a_2, a_3)) < \delta$.

Then $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1, a_2, a_3)) = \delta_1 < \delta$. If $\delta_1 = 0$, we are done.

If $\delta_1 > 0$, then by the definition of $\text{ORD}_1^{3,D}$,

$$\forall a_4 \forall a_5 \exists a_6 \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1, a_2, L, a_5, a_6)) \downarrow \text{ and is } < \delta_1.$$

Pick any a_4 , pick any a_5 , and select an a_6 that satisfies

$\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1, a_2, L, a_5, a_6)) < \delta_1$. Then $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1, a_2, L, a_5, a_6)) = \delta_2 < \delta_1$. If

$\delta_2 = 0$, we are done.

If $\delta_2 > 0$, we continue in this manner unless/until we reach $\delta_n = 0$. Since decreasing sequence $\delta > \delta_1 > \delta_2 > L$ of ordinals is finite, eventually we do have $\delta_n = 0$. Thus we obtain $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}(a_1, a_2, a_3, L, a_{3n-1}, a_{3n})) = 0$ and so $a_1 = M$. This is a contradiction of $a_1 \neq M$. Thus if $\ln(\overset{\mathbf{r}}{p}) \leq 1$, $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{p}) \uparrow$. (Lemma 4.2)

Claim 4.3. $\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) \downarrow$ iff $\ln(\overset{\Gamma}{p}) \geq 2$ and $a_1(\overset{\Gamma}{p}) = M$.

(\Leftarrow) First show if the $\ln(\overset{\Gamma}{p}) \geq 2$ and $a_1(\overset{\Gamma}{p}) = M$, then $\overset{\Gamma}{p}$ has $\text{ORD}_1^{3,D}$.

Suppose $a_1(\overset{\Gamma}{p}) = M$, i.e. $\overset{\Gamma}{p} = (a_0, M, a_2, \mathbf{K}, a_{n-1}, a_n)$ for some $a_0, a_2, a_3, \mathbf{K}, a_{n-1}, a_n$. By the definition of ordinals of a position, $\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) = 0$ so that $\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) \downarrow$.

(\Rightarrow) Next we show, if $\overset{\Gamma}{p}$ has $\text{ORD}_1^{3,D}$, then $\ln(\overset{\Gamma}{p}) \geq 2$ and $a_1(\overset{\Gamma}{p}) = M$.

Suppose $\overset{\Gamma}{p} = (a_0, a_1, \mathbf{K}, a_{n-1}, a_n)$ has $\text{ORD}_1^{3,D}$. From Lemma 4.2 we have $\ln(\overset{\Gamma}{p}) \geq 2$ and $n \geq 1$. Let $\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) = \beta$, and show $a_1(\overset{\Gamma}{p}) = M$.¹ We show this by induction on β .

For the Base Step, $\beta = 0$ so that $\text{ORD}_1^{3,D}(a_0, a_1, \mathbf{K}, a_{n-1}, a_n) = 0$, and by the definition of $\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) = 0$, $a_1 = M$.

For the Induction Step we assume $\beta > 0$ and the Induction Hypothesis that:

for any $\overset{\Gamma}{q}' = (a_0, a_1, \mathbf{K}, a_{j-1}, a_j)$ in which $\text{ORD}_1^{3,D}(\overset{\Gamma}{q}') < \beta$,

we have $j \geq 1$ and $a_1(\overset{\Gamma}{q}') = M$.

Now we must show $a_1(\overset{\Gamma}{p}) = M$, knowing that $\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) = \beta$. First we shall find

$\overset{\Gamma}{q} \supseteq (a_0, a_1, a_2, \mathbf{L}, a_{n-1}, a_n)$ such that $\text{ORD}_1^{3,D}(\overset{\Gamma}{q}) \downarrow$ and is $< \beta$.

Case 1: $n = 3k$ for some $k \geq 0$.

Recall $\text{ORD}_1^{3,D}(a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) = \beta > 0$. By the definition of $\text{ORD}_1^{3,D}$,

$$\forall x_1 \forall x_2 \exists x_3 \text{ORD}_1^{3,D}(\overset{\Gamma}{p}(x_1, x_2, x_3)) < \beta. \quad (1)$$

Since $\beta > 0$, pick any $x_1 = a_{3k+1}$, pick any $x_2 = a_{3k+2}$, and select an $x_3 = a_{3k+3}$ that makes

¹ Therefore, $\text{ORD}_1^{3,D}(\overset{\Gamma}{p}) = 0$.

(1) true. Then $\text{ORD}_I^{3,D}(\overset{\Gamma}{p})(a_{3k+1}, a_{3k+2}, a_{3k+3}) < \beta$. We let $\overset{\Gamma}{q} =_{\text{df}} \overset{\Gamma}{p}(a_{3k+1}, a_{3k+2}, a_{3k+3})$.

We have $\overset{\Gamma}{q} \supseteq \overset{\Gamma}{p}$, $\text{ORD}_I^{3,D}(\overset{\Gamma}{q}) \downarrow$, and $\text{ORD}_I^{3,D}(\overset{\Gamma}{q}) < \beta$.

Case 2: $n = 3k + 1$ for some $k \geq 0$.

Recall $\text{ORD}_I^{3,D}(a_0, a_1, \mathbf{K}, a_{3k}, a_{3k+1}) = \beta$. By the definition of $\text{ORD}_I^{3,D}$,

$$\forall x_2 \exists x_3 \text{ORD}_I^{3,D}(\overset{\Gamma}{p}(x_1, x_2)) < \beta. \quad (2)$$

Since $\beta > 0$, pick any $x_2 = a_{3k+2}$, and select an $x_3 = a_{3k+3}$ that makes (2) true. Then

$\text{ORD}_I^{3,D}(\overset{\Gamma}{p}(a_{3k+2}, a_{3k+3})) < \beta$. We let $\overset{\Gamma}{q} =_{\text{df}} \overset{\Gamma}{p}(a_{3k+2}, a_{3k+3})$. We have $\overset{\Gamma}{q} \supseteq \overset{\Gamma}{p}$,

$\text{ORD}_I^{3,D}(\overset{\Gamma}{q}) \downarrow$, and $\text{ORD}_I^{3,D}(\overset{\Gamma}{q}) < \beta$.

Case 3: $n = 3k + 2$ for some $k \geq 0$.

Recall $\text{ORD}_I^{3,D}(a_0, a_1, \mathbf{K}, a_{3k+1}, a_{3k+2}) = \beta$. By the definition of $\text{ORD}_I^{3,D}$,

$$\exists x_3 \text{ORD}_I^{3,D}(\overset{\Gamma}{p}(x_3)) < \beta. \quad (3)$$

Since $\beta > 0$, select an $x_3 = a_{3k+3}$ that makes (3) true. Then $\text{ORD}_I^{3,D}(\overset{\Gamma}{p}(a_{3k+3})) < \beta$. We

let $\overset{\Gamma}{q} =_{\text{df}} \overset{\Gamma}{p}(a_{3k+3})$, $\overset{\Gamma}{q} \supseteq \overset{\Gamma}{p}$, $\text{ORD}_I^{3,D}(\overset{\Gamma}{q}) \downarrow$, and is $< \beta$.

Thus by Cases 1, 2, and 3, we have $\overset{\Gamma}{q} \supseteq (a_0, a_1, \mathbf{K}, a_{n-1}, a_n)$, $\text{ORD}_I^{3,D}(\overset{\Gamma}{q}) \downarrow$, and is

$< \beta$. By the Induction Hypothesis, $a_1(\overset{\Gamma}{q}) = M$. Since $\text{ln}(\overset{\Gamma}{p}) \geq 2$ and $\overset{\Gamma}{q} \supseteq \overset{\Gamma}{p}$, $a_1(\overset{\Gamma}{p})$

exists and equals $a_1(\overset{\Gamma}{q}) = M$. , (Claim 4.3)

Next we compute T^{LD} and $\text{ORD}_{\text{IE}}^{3,\text{T}^{\text{LD}}}$ for the game. Recall from Chapter 3,

$\text{T}^{\text{LD}} =_{\text{df}} \{ \overset{\Gamma}{p} = (a_0, a_1, \mathbf{K}, a_{n-1}) \uparrow \text{ORD}_I^{3,D}(\overset{\Gamma}{p}) \uparrow \}$. By Claim 4.3,

$\text{T}^{\text{LD}} = \{ \overset{\Gamma}{p} = (a_0, a_1, \mathbf{K}, a_{n-1}) \uparrow \text{an}(\overset{\Gamma}{p}) \leq 1 \text{ or } a_1(\overset{\Gamma}{p}) \neq M \}$.

Since the payoff set for player II is empty, no $\overset{\text{I}}{p}$ has $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}$ -value of zero (by definition of $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}$ 0). By induction on β , one can prove that no $\overset{\text{I}}{p}$ has $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}$ -value β . Thus $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) \uparrow$ for every $\overset{\text{I}}{p}$, so that:

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}} = \emptyset. \quad (1)$$

We now show that G satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\emptyset]{\text{T}^{\text{LD}} \text{ help}} \text{II}$.

Claim 4.4. G satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$.

Suppose $\overset{\text{I}}{p} = (a_0, a_1, a_2, L, a_{3k}, a_{3k+1}) \notin \text{ORD}_1^{3,\text{D}}$ and $\exists y$ such that $\overset{\text{I}}{p}(y) \notin \text{ORD}_1^{3,\text{D}}$. We shall show that player III must play a_{3k+2} such that $\overset{\text{I}}{p}(a_{3k+2}) \notin \text{ORD}_1^{3,\text{D}}$. We know $\ln(\overset{\text{I}}{p}) \geq 1$. By Claim 4.3, $a_1(\overset{\text{I}}{p}) \neq M$ so that $\forall a_{3k+2} a_1(\overset{\text{I}}{p}(a_{3k+2})) \neq M$. Therefore $\forall a_{3k+2} \overset{\text{I}}{p}(a_{3k+2}) \notin \text{ORD}_1^{3,\text{D}}$ by Claim 1 again. Thus player III can pick any a_{3k+2} and player I will not be helped. , (Claim 4.4)

Claim 4.5. G satisfies $\text{I} \xrightarrow[\emptyset]{\text{T}^{\text{LD}} \text{ help}} \text{II}$.

Suppose $\overset{\text{I}}{p} = (a_0, a_1, K, a_{3k-2}, a_{3k-1}) \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II},\emptyset}^{3,\text{T}^{\text{LD}}}$ and $\exists y$ such that $\overset{\text{I}}{p}(y) \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II},\emptyset}^{3,\text{T}^{\text{LD}}}$. We shall show player I must play a_{3k} such that $\overset{\text{I}}{p}(a_{3k}) \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II},\emptyset}^{3,\text{T}^{\text{LD}}}$. Since $\text{ORD}_{\text{II}}^{3,\text{T}^{\text{I}}} = \emptyset$, by (i), $\text{T}^{\text{LD}} \setminus \text{ORD}_{\text{II}}^{3,\text{T}^{\text{I}}} = \text{T}^{\text{LD}}$.

Case 1: $k = 0$.

Then $\overset{\text{I}}{p} = (\langle \rangle)$ and player I selects any a_0 . Since $\ln(\overset{\text{I}}{p}(a_0)) \leq 1$, $\overset{\text{I}}{p}(a_0) \in \text{T}^{\text{LD}}$ by

Lemma 4.2.

Case 2: $k \geq 1$.

Then $\ln(\overset{\mathbf{r}}{p}) \geq 2$, and by the definition of $T^{1,D}$ and by Claim 4.3, $a_1(\overset{\mathbf{r}}{p}) \neq M$ since $\overset{\mathbf{r}}{p} \in T^{1,D}$. Let player I select any a_{3k} . Since $a_1(\overset{\mathbf{r}}{p}(a_{3k})) = a_1(\overset{\mathbf{r}}{p}) \neq M$, $\overset{\mathbf{r}}{p}(a_{3k}) \in T^{1,D}$.

In both Cases 1 and 2 $\forall a_{3k} \overset{\mathbf{r}}{p}(a_{3k}) \in T^{1,D} = T^{1,D} \setminus \text{ORD}_{\text{II}, \emptyset}^{3, T^{1,D}}$. Thus player I can choose any a_{3k} and player II will not be helped. , (Claim 4.5)

By Claims 4.4 and 4.5, G satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\emptyset]{T^{1,D} \text{ help}} \text{II}$.

Claim 4.6 The game $G = G_M^A(O(D), \emptyset, \vec{O}(D))$ is not determined.

Proof. Let σ be a strategy for the game presented here. We will show σ is not a winning strategy for any player.

Case 1: Assume σ is a strategy for player I.

Let $\overset{\mathbf{r}}{y} = (a_0, a_1, K, a_{n-1}, a_n, K)$ be a legal play according to σ , in which player II selects $a_1 \neq M$. Since $a_1 \neq M$, $\overset{\mathbf{r}}{y}$ is not a win for player I. Therefore, since $\overset{\mathbf{r}}{y}$ is according to σ , σ cannot be a winning strategy for player I.

Case 2: Assume σ is a strategy for player II.

Let $\overset{\mathbf{r}}{y} = (a_0, a_1, K, a_{n-1}, a_n, K)$ be a legal play according to σ . Since player II has an empty payoff set, $\overset{\mathbf{r}}{y}$ is not a win for player II. Therefore, since $\overset{\mathbf{r}}{y}$ is according to σ , σ cannot be a winning strategy for player II.

Case 3: Assume σ is a strategy for player III.

Let $\overset{r}{y} = (a_0, a_1, K, a_{n-1}, a_n, K)$ be a legal play according to σ , in which player II selects $a_1 = M$. Since $a_1 = M$, $\overset{r}{y}$ is not a win for player III. Therefore, since $\overset{r}{y}$ is according to σ , σ cannot be a winning strategy for player III. Thus σ is not a winning strategy.

Thus, by Cases 1, 2 and 3, the game G is not determined.

, (Claim 4.6 and Theorem 4.1)

Next we shall define a nondetermined three-player game G in which players I and II have open payoff sets and which satisfies $\text{II} \not\rightarrow_{\text{D}}^{\text{help}} \text{I}$ and $\text{I} \not\rightarrow_{\text{E}}^{\text{T}^{\text{LD}} \text{help}} \text{II}$. By Theorem 3.19, G cannot satisfy $\text{III} \not\rightarrow_{\text{D}}^{\text{help}} \text{I}$ since otherwise G would be determined.

Theorem 4.7. Theorem 3.19 with $\text{III} \not\rightarrow_{\text{D}}^{\text{help}} \text{I}$ omitted is false, i.e. there exists a nondetermined game G in which the payoff sets of players I and II are respectively open sets $O(D)$ and $O(E)$ and which satisfies $\text{II} \not\rightarrow_{\text{D}}^{\text{help}} \text{I}$ and $\text{I} \not\rightarrow_{\text{E}}^{\text{T}^{\text{LD}} \text{help}} \text{II}$.

Proof. Let A be any set with at least two elements and let $M \in A$. We shall present a three player (nondetermined) game $G = G_M^A(O(D), O(E), \emptyset)$ that satisfies the above theorem:

$$G: \begin{array}{cccccccc} & \text{I} & a_0 & & a_3 & & a_{3n} & \\ & \text{II} & & a_1 & & a_4 & \text{L} & a_{3n+1} & \text{L} & . \\ & \text{III} & & & a_2 & & a_5 & & a_{3n+2} & \end{array}$$

Player I wins iff $a_2 = M$. Player II wins iff $a_2 \neq M$. Player III never wins and player III's payoff set is the empty set. Below we define D and E , and in Claims 4.9, 4.10, and 4.11 we respectively prove that G satisfies $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}} \text{ help}} \text{II}$, and that G is not determined.

Let D be the collection of all finite sequences of the form (a_0, a_1, M) where $a_0, a_1 \in A$. Let E be the collection of all finite sequences of the form (a_0, a_1, a_2, \dots) where $a_0, a_1, a_2 \in A$ and $a_2 \neq M$. Let $a_2(\vec{p})$ denote a_2 when $\vec{p} = (a_0, a_1, a_2, \dots, a_{n-1}, a_n)$. Player I and II's payoff sets are the open sets $O(D)$ and $O(E)$ respectively. Note that:

$$f \in O(D) \text{ iff } \exists a_0 \exists a_1 (a_0, a_1, M) \subseteq f.$$

$$\text{ORD}_I^{3,D}(\vec{p}) = 0 \text{ iff } a_2(\vec{p}) = M \text{ for some } a_0, a_1.$$

$$\text{ORD}_I^{3,D}(\vec{p}) = 0 \text{ iff } \text{ln}(\vec{p}) \geq 3 \text{ and } a_2(\vec{p}) = M.$$

$$h \in O(E) \text{ iff } \exists a_0 \exists a_1 \exists a_2 (a_2 \neq M \wedge (a_0, a_1, a_2) \subseteq h).$$

$$\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(\vec{p}) = 0 \text{ iff } a_2(\vec{p}) \neq M \text{ for some } a_0, a_1.$$

Lemma 4.8. If $\text{ln}(\vec{p}) \leq 2$, then $\text{ORD}_I^{3,D}(\vec{p}) \uparrow$, i.e. \vec{p} has no $\text{ORD}_I^{3,D}$ -value.

This proof is analogous to the proof of Lemma 4.2 in Theorem 4.1. Assume $\text{ln}(\vec{p}) \leq 2$ and $\text{ORD}_I^{3,D}(\vec{p}) \downarrow$ (towards a contradiction). First find $\vec{q} \supseteq \vec{p}$ such that $\text{ln}(\vec{q}) = 2$ and $\text{ORD}_I^{3,D}(\vec{q}) \downarrow$. By the definition of $\text{ORD}_I^{3,D}0$ and $\text{ORD}_I^{3,D}$, we have

$$\delta = \text{ORD}_I^{3,D}(\vec{q}) > 0,$$

$$\forall a_2 \text{ ORD}_I^{3,D}(\vec{q}(a_2)) \downarrow \text{ and } \leq \delta,$$

$$\forall a_2 \exists a_3 \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q})(a_2, a_3) \downarrow \text{ and } < \delta.$$

Pick any $a_2 \neq M$, and select an a_3 that satisfies $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q})(a_2, a_3) = \delta_1 < \delta$.

By the usual construction, $\exists \vartheta \subseteq \overset{\mathbf{r}}{q}(a_2, a_3)$ such that $\text{ORD}_1^{3,D}(\vartheta) = 0$. Since $\text{ORD}_1^{3,D}(\vartheta) = 0$, $a_2(\vartheta) = M$, $\rightarrow \leftarrow$ of $a_2(\vartheta) = a_2 \neq M$. , (Lemma 4.8)

Claim 4.9. $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{p}) \downarrow$ iff $\ln(\overset{\mathbf{r}}{p}) \geq 3$ and $a_2(\overset{\mathbf{r}}{p}) = M$.

This proof of Claim 4.9 is analogous to the proof of Claim 4.3 in Theorem 4.1.

(\Leftarrow) If $a_2(\overset{\mathbf{r}}{p})$ exists and equals M , then $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{p}) = 0$. Therefore an $\text{ORD}_1^{3,D}$ -value exists.

(\Rightarrow) Next, show that if $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{p}) \downarrow$, then the $\ln(\overset{\mathbf{r}}{p}) \geq 3$ and $a_2(\overset{\mathbf{r}}{p}) = M$.

Suppose $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{p}) \downarrow$. Then by the Lemma 4.2 $\ln(\overset{\mathbf{r}}{p}) \geq 3$. Just as is the proof of Claim 1 of Theorem 4.1, one can prove by induction on β , that if $\beta = \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}) \downarrow$, then $a_2(\overset{\mathbf{r}}{q}) = M$. Then it will follow that $a_2(\overset{\mathbf{r}}{p}) = M$ since $\text{ORD}_1^{3,D}(\overset{\mathbf{r}}{p}) \downarrow$.

, (Claim 4.9)

As usual, let $T^{\text{LD}} =_{\text{df}} \left\{ \overset{\mathbf{r}}{p} \mid \forall \overset{\mathbf{r}}{q} \subseteq \overset{\mathbf{r}}{p} \text{ORD}_1^{3,D}(\overset{\mathbf{r}}{q}) \uparrow \right\}$. By Claim 4.9,

$$T^{\text{LD}} = \left\{ \overset{\mathbf{r}}{p} = (a_0, a_1, a_2, L, a_{n-1}) \mid \ln(\overset{\mathbf{r}}{p}) \leq 2 \text{ or } a_2(\overset{\mathbf{r}}{p}) \neq M \right\}. \quad (\$)$$

Recall player Π 's payoff set is the open set $O(E)$ and E is the collection of all finite sequences of the form (a_0, a_1, a_2) where $a_0, a_1, a_2 \in A$ and $a_2 \neq M$. Also:

$$h \in O(E) \text{ iff } \exists a_0 \exists a_1 \exists a_2 (a_2 \neq M \wedge (a_0, a_1, a_2) \subseteq h).$$

Remark 1: By definition, not only is $\frac{\mathbf{r}}{p} \in T^{\text{LD}}$ when both $\ln(\frac{\mathbf{r}}{p}) \geq 3$ and $a_2(\frac{\mathbf{r}}{p}) \neq M$, but in fact:

$$\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(\frac{\mathbf{r}}{p}) = 0 \text{ iff } \ln(\frac{\mathbf{r}}{p}) \geq 3 \text{ and } a_2(\frac{\mathbf{r}}{p}) \neq M.$$

Since $\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(\frac{\mathbf{r}}{p}) \downarrow$ implies $\frac{\mathbf{r}}{p} \in T^{\text{LD}}$ and since $(\ln(\frac{\mathbf{r}}{p}) \geq 3 \wedge \frac{\mathbf{r}}{p} \in T^{\text{LD}}) \Rightarrow a_2(\frac{\mathbf{r}}{p}) \neq M$, by

(\\$) we have:

Remark 2: If $\ln(\frac{\mathbf{r}}{p}) \geq 3$, then $\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(\frac{\mathbf{r}}{p}) \downarrow \Leftrightarrow a_2(\frac{\mathbf{r}}{p}) \neq M \Leftrightarrow \text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(\frac{\mathbf{r}}{p}) = 0$.

Claim 4.10. $\forall a_0 \forall a_1 \text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(a_0, a_1) = 1$.

Pick any a_0 and a_1 . We will show $\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(a_0, a_1) = 1$. By the definition of $\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}$, we must show $(a_0, a_1) \in T^{\text{LD}}$ and $\forall a_2 \in T_{(a_0, a_1)}^{\text{LD}}, \text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(a_0, a_1, a_2) \leq 1$. The former, $(a_0, a_1) \in T^{\text{LD}}$, follows by Lemma 4.1. To show the latter, pick any $a_2 \in T_{(a_0, a_1)}^{\text{LD}}$ and show $\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(a_0, a_1, a_2) = 0$ which is certainly ≤ 1 . Since $a_2 \in T_{(a_0, a_1)}^{\text{LD}}$, $(a_0, a_1, a_2) \in T^{\text{LD}}$ and $a_2 \neq M$ by (\\$). Therefore $\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(a_0, a_1, a_2) = 0$.

, (Claim 4.10)

Claim 4.11. $\forall a_0 \text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(a_0) = 2$.

Pick any a_0 . By Lemma 4.1, $a_0 \in T^{\text{LD}}$. To show $\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(a_0) = 2$, show $\exists a_1$ such that $(a_0, a_1) \in T^{\text{LD}}$ and $\text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}(a_0, a_1) \leq 1$. Claim 4.10 gives us this for any a_1 .

, (Claim 4.11)

Claim 4.12. $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(\langle \rangle) = 2$.

By Lemma 4.8, $\langle \rangle \in \text{T}^{\text{LD}}$. We need to show $\forall (a_0) \in \text{T}^{\text{LD}}, \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(a_0) \leq 2$. This follows immediately from Claim 4.11. , (Claim 4.12)

Remark 3. $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(\langle \rangle) \downarrow$ and is equal to 2, even though player II has no winning strategy and in particular $\text{ORD}_{\text{II}}^{3,\text{E}}(\langle \rangle) \uparrow$.

Claim 4.13. $\text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}(\overset{\text{r}}{p}) \downarrow$ iff $\ln(\overset{\text{r}}{p}) \leq 2$ or $a_2(\overset{\text{r}}{p}) \neq M$. Hence $\text{T}^{\text{LD}} = \text{ORD}_{\text{II,E}}^{3,\text{T}^{\text{LD}}}$ by (\$).

The direction \Leftarrow follows by Claims 4.10-4.12 when $\ln(\overset{\text{r}}{p}) \leq 2$ and by the \Leftarrow direction of Remark 1 when $\ln(\overset{\text{r}}{p}) \geq 3$.

The direction \Rightarrow follows by Remark 2. , (Claim 4.13)

We now show that G satisfies $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$ and $\text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}} \text{ help}} \text{II}$ so that in fact followed players I and II cannot help each other.

Claim 4.14. G satisfies $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$.

Suppose $\overset{\text{r}}{p} = (a_0, a_1, K, a_{3k-1}, a_{3k}) \notin \text{ORD}_{\text{I}}^{3,\text{D}}$ and $\exists y$ such that $\overset{\text{r}}{p}(y) \notin \text{ORD}_{\text{I}}^{3,\text{D}}$. We shall show that player II will play a_{3k+1} such that $\overset{\text{r}}{p}(a_{3k+1}) \notin \text{ORD}_{\text{I}}^{3,\text{D}}$.

Case 1: $k = 0$.

Then $\overset{\text{r}}{p} = (a_0)$. Let player II select any a_1 . Since $\ln(\overset{\text{r}}{p}(a_1)) \leq 2$, $\overset{\text{r}}{p}(a_1) \notin \text{ORD}_{\text{I}}^{3,\text{D}}$

by Lemma 4.8.

Case 2: $k \geq 1$.

Then $\ln(\overset{\mathbf{r}}{p}) \geq 3$ and $a_2(\overset{\mathbf{r}}{p}) \neq M$ since $\overset{\mathbf{r}}{p} \notin \text{ORD}_I^{3,D}$. Let player II select any a_{3k+1} .

Then $a_2(\overset{\mathbf{r}}{p}(a_{3k+1})) = a_2(\overset{\mathbf{r}}{p}) \neq M$.

In both cases (i) and (ii) $\forall a_{3k+1} \overset{\mathbf{r}}{p}(a_{3k+1}) \notin \text{ORD}_I^{3,D}$. Thus player II can choose any a_{3k+1} and player I will not be helped. , (Claim 4.14)

Claim 4.15. G satisfies $I \xrightarrow[\cancel{E}]{\cancel{\text{help}}} \text{II}$.

We show that if $\overset{\mathbf{r}}{p} = (a_0, a_1, a_2, L, a_{3k-2}, a_{3k-1}) \in T^{\text{LD}} \setminus \text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}$ and if $\exists y$ such that $\overset{\mathbf{r}}{p}(y) \in T^{\text{LD}} \setminus \text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}$, then player I must play a_{3k} such that $\overset{\mathbf{r}}{p}(a_{3k}) \in T^{\text{LD}} \setminus \text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}}$. By Claim 5, $T^{\text{LD}} \setminus \text{ORD}_{\text{II,E}}^{3,T^{\text{LD}}} = \emptyset$ so that the antecedent (of what we need to show) is always false. Therefore player I cannot help player II. , (Claim 4.15)

By Claims 4.14 and 4.15, G satisfies $\text{II} \xrightarrow[\cancel{D}]{\cancel{\text{help}}} \text{I}$ and $\text{I} \xrightarrow[\cancel{E}]{\cancel{\text{help}}} \text{II}$.

Claim 4.16. The game $G = G_M^A(O(D), O(E), \emptyset)$ is not determined.

The proof of this claim is analogous to the proof of Claim 4.6 in Theorem 4.1.

Let σ be a strategy for this game. We shall select a legal play

$\overset{\mathbf{r}}{y} = (a_0, a_1, K, a_{n-1}, a_n, K)$ that is according to σ , but witnesses that σ is not a winning strategy. If σ is a strategy for player I, we pick $\overset{\mathbf{r}}{y}$ according to σ in which $a_2(\overset{\mathbf{r}}{p}) \neq M$.

This results in \hat{y} is not a win for player I, so σ is not a winning strategy for player I. If σ is a strategy for player II, we pick \hat{y} according to σ in which $a_2(\hat{p}) = M$. This results in \hat{y} is not a win for player II, so σ is not a winning strategy for player II. Since player III has empty payoff set, σ cannot be a winning strategy for player III. Thus σ is not a winning strategy.

Therefore the game presented here is not determined.

, (Claim 4.16 and Theorem 4.7)

Next we shall define a nondetermined three player game G in which player I has the open payoff set $\emptyset = O(\emptyset)$, player II has an open payoff set, and G satisfies $\text{III} \xrightarrow[\emptyset]{\text{help}} \text{I}$ and $\text{II} \xrightarrow[\emptyset]{\text{help}} \text{I}$. By Theorem 3.1, G cannot satisfy $\text{I} \xrightarrow[E]{\text{T}^{\text{LD}} \text{help}} \text{II}$ for any E that generates player II's open payoff set since otherwise G would be determined.

Theorem 4.17. Theorem 3.19 with $\text{I} \xrightarrow[E]{\text{T}^{\text{LD}} \text{help}} \text{II}$ omitted is false, i.e. there exists a nondetermined game G in which the payoff sets of players I and II are respectively open sets $\emptyset = O(\emptyset)$ and $O(E)$ and which satisfies $\text{III} \xrightarrow[\emptyset]{\text{help}} \text{I}$ and $\text{II} \xrightarrow[\emptyset]{\text{help}} \text{I}$.

Proof. Let A be any set with at least two elements and let $M \in A$.

We shall present a three player (nondetermined) game $G = G_M^A(\emptyset, O(E), \vec{O}(E))$ that satisfies the above theorem:

$$\begin{array}{ccccccc}
& \text{I} & a_0 & & a_3 & & a_{3n} \\
G: & \text{II} & & a_1 & & a_4 & \text{L} & & a_{3n+1} & & \text{L} & . \\
& \text{III} & & & a_2 & & & a_5 & & & & a_{3n+2}
\end{array}$$

Player I never wins and player I's payoff set is the open set $O(\emptyset)$, which is just the empty set. Player II wins iff $a_0 = M$. Player III wins iff $a_0 \neq M$. Player II's payoff set is the open set $O(E)$. Below we define E and in Claims 2 and 3, we respectively prove that G satisfies $\text{III} \xrightarrow[\emptyset]{\text{help}} \text{I}$ and $\text{II} \xrightarrow[\emptyset]{\text{help}} \text{I}$ and that G is not determined.

Since the payoff set for player I is empty, no $\overset{\text{I}}{p}$ has $\text{ORD}_1^{3,\emptyset} 0$ (by definition of $\text{ORD}_1^{3,\emptyset} 0$). By induction on β , one can prove that no $\overset{\text{I}}{p}$ has $\text{ORD}_1^{3,\emptyset}$ of value β . Thus $\text{ORD}_1^{3,\emptyset}(\overset{\text{I}}{p}) \uparrow$ for every $\overset{\text{I}}{p}$, and the $\text{ORD}_1^{3,\emptyset} = \emptyset$. Recall

$\text{T}^{\text{LD}} =_{\text{df}} \left\{ \overset{\text{I}}{p} \mid \forall \overset{\text{I}}{q} \subseteq \overset{\text{I}}{p} \text{ ORD}_1^{3,\text{D}}(\overset{\text{I}}{q}) \uparrow \right\}$. Since $\text{ORD}_1^{3,\text{D}} = \emptyset$, T^{LD} equals the set of all finite sequences $\overset{\text{I}}{p}$.

Let E be the collection of all finite sequences of the form (M) . Note that:

$$h \in O(E) \text{ iff } (M) \subseteq h.$$

$$\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0 \text{ iff } \overset{\text{I}}{p} \supseteq (M), \text{ i.e. } \text{ORD}_{\text{II}}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = 0 \text{ iff } a_0(\overset{\text{I}}{p}) = M.$$

Since $\text{T}^{\text{LD}} = A^{<\omega}$, $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}(\overset{\text{I}}{p}) = \text{ORD}_{\text{II}}^{3,E}(\overset{\text{I}}{p}) = 0$ iff $a_0(\overset{\text{I}}{p}) = M$; therefore our restricted ordinal-values $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}$ is the same as the non-restricted version $\text{ORD}_{\text{II}}^{3,E}$.

Lemma 4.18. $\text{ORD}_{\text{II}}^{3,E}(\langle \rangle) \uparrow$ and $\text{ORD}_{\text{II},E}^{3,\text{T}^{\text{LD}}}(\langle \rangle) \uparrow$.

This proof is analogous to the proof of Lemma 4.2 in Theorem 4.1 for $\text{ORD}_1^{3,\text{D}}$.

Assume $\text{ORD}_{\text{II}}^{3,E}(\langle \rangle) \downarrow$ or $\text{ORD}_{\text{II,E}}^{3,T^1,\emptyset}(\langle \rangle) \downarrow$ (towards a contradiction). Then

$\text{ORD}_{\text{II}}^{3,E}(\langle \rangle) \downarrow$ and $\text{ORD}_{\text{II,E}}^{3,T^1,\emptyset}(\langle \rangle) \downarrow$ and $\text{ORD}_{\text{II}}^{3,E}(\langle \rangle) = \text{ORD}_{\text{II,E}}^{3,T^1,\emptyset}(\langle \rangle)$. By the definition of $\text{ORD}_{\text{II}}^{3,E}$, we have

$$\forall a_0 \text{ORD}_{\text{II}}^{3,E}(a_0) \downarrow \text{ and } \leq \text{ORD}_{\text{II}}^{3,E}(\langle \rangle).$$

Pick any $a_0 \neq M$. Then $\text{ORD}_{\text{II}}^{3,E}(a_0) \downarrow$.

By the usual construction, $\exists \overset{\text{r}}{p} \supseteq (a_0)$ such that $\text{ORD}_{\text{II}}^{3,E}(\overset{\text{r}}{p}) = 0$. Since

$$\text{ORD}_{\text{II}}^{3,E}(\overset{\text{r}}{p}) = 0, a_0(\overset{\text{r}}{p}) = M, \rightarrow \leftarrow \text{ of } a_0(\overset{\text{r}}{p}) \neq M. \quad , \text{ (Lemma 4.18)}$$

Claim 4.19. $\text{ORD}_{\text{II}}^{3,E}(\overset{\text{r}}{p}) \downarrow$ iff $\ln(\overset{\text{r}}{p}) \geq 1$ and $a_0(\overset{\text{r}}{p}) = M$.

This proof of Claim 4.19 is analogous to the proof of Claim 4.3 in Theorem 4.1 for $\text{ORD}_1^{3,D}$.

(\Leftarrow) If $a_0(\overset{\text{r}}{p})$ exists and equals M , then $\text{ORD}_{\text{II}}^{3,E}(\overset{\text{r}}{p}) = 0$ and therefore the ordinal exists.

(\Rightarrow) Suppose $\text{ORD}_{\text{II}}^{3,E}(\overset{\text{r}}{p}) \downarrow$. Then by Lemma 4.18 $\ln(\overset{\text{r}}{p}) \geq 1$. We show $a_0(\overset{\text{r}}{p}) = M$.

Construct $\overset{\text{r}}{q} \supseteq \overset{\text{r}}{p}$ such that $\text{ORD}_{\text{II}}^{3,E}(\overset{\text{r}}{q}) = 0$. Then $a_0(\overset{\text{r}}{q}) = M$. Since $\overset{\text{r}}{q} \supseteq \overset{\text{r}}{p}$,

$$a_0(\overset{\text{r}}{p}) = a_0(\overset{\text{r}}{q}) = M. \quad , \text{ (Claim 4.19)}$$

We now show that G satisfies $\text{III} \xrightarrow[\emptyset]{\text{help}} \text{I}$ and $\text{II} \xrightarrow[\emptyset]{\text{help}} \text{I}$.

Claim 4.20. G satisfies $\text{III} \xrightarrow[\emptyset]{\text{help}} \text{I}$ and $\text{II} \xrightarrow[\emptyset]{\text{help}} \text{I}$.

Suppose $\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \notin \text{ORD}_I$ and $\exists y$ such that $\overset{\mathbf{r}}{p}(y) \notin \text{ORD}_I^{3, \emptyset}$. We shall show that for any a_{3k+1} and a_{3k+2} respectively played by players II and III, such that neither $\overset{\mathbf{r}}{p}(a_{3k+1}) \notin \text{ORD}_I^{3, \emptyset}$ nor $\overset{\mathbf{r}}{p}(a_{3k+1}, a_{3k+2}) \notin \text{ORD}_I^{3, \emptyset}$. Since $\overset{\mathbf{r}}{p} = (a_0, a_1, \mathbf{K}, a_{3k-1}, a_{3k}) \notin \text{ORD}_I^{3, \emptyset}$, then for any a_{3k+1} played by player II $\overset{\mathbf{r}}{p}(a_{3k+1}) \notin \text{ORD}_I^{3, \emptyset}$. Since $\overset{\mathbf{r}}{p}(a_{3k+1}) \notin \text{ORD}_I^{3, \emptyset}$, then for any a_{3k+2} played by player III $\overset{\mathbf{r}}{p}(a_{3k+1}, a_{3k+2}) \notin \text{ORD}_I^{3, \emptyset}$. Thus for any a_{3k+1} and a_{3k+2} played by players II and III, respectively, $\overset{\mathbf{r}}{p}(a_{3k+1}) \notin \text{ORD}_I^{3, \emptyset}$ and $\overset{\mathbf{r}}{p}(a_{3k+1}, a_{3k+2}) \notin \text{ORD}_I^{3, \emptyset}$. , (Claim 4.20)

Claim 4.21. The game $G = G_M^A(\emptyset, O(E), \vec{O}(E))$ is not determined.

Let σ be a strategy for this game. We shall select a legal play $\overset{\mathbf{r}}{y} = (a_0, a_1, \mathbf{K}, a_{n-1}, a_n, \mathbf{K})$ that is according to σ , but witnesses that σ is not a winning strategy. Since player I has an empty payoff set, σ cannot be a winning strategy for player I. If σ is a strategy for player II, we pick $\overset{\mathbf{l}}{y}$ according to σ in which $a_0(\overset{\mathbf{r}}{p}) = M$. This results in $\overset{\mathbf{l}}{y}$ is not a win for player II and σ is not a winning strategy for player II. If σ is a strategy for player III, we pick $\overset{\mathbf{l}}{y}$ according to σ in which $a_0(\overset{\mathbf{r}}{p}) \neq M$. This results in $\overset{\mathbf{l}}{y}$ is not a win for player III and σ is not a winning strategy for player III. Thus, σ is not a winning strategy.

Consequently, the game presented here is not determined.

, (Claim 4.21 and Theorem 4.17)

By Theorems 4.1, 4.7 and 4.17, we have shown that it is impossible to prove any

version of Theorem 3.19 which omits at least one of the non-helping hypothesis,

$$\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}, \text{ II} \xrightarrow[\text{D}]{\text{help}} \text{I}, \text{ and } \text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}} \text{ help}} \text{II}.$$

CHAPTER 5

PERMUTING THE PLAYER'S ROLES

The proof presented in Chapter 3 had assigned players I and II with open payoff sets. In this chapter we will verify that assigning the two open payoff sets to other players does not change the determinacy of the games as long as the non-helping conditions are also adapted to reflect those changes. The non-helping conditions presented in our “permuted” games will follow those conditions stated in the proof of Chapter 3, i.e. the players with the open payoff sets do not help each other and the player with the complementary payoff set does not help the player with the “primary” open payoff set. Since the payoff sets will be assigned to other players, the definitions of the ordinals of positions for those players will also be changed.

In the main theorem of Chapter 3, players I and II were assigned open payoff sets $O(D)$ and $O(E)$, and $\text{ORD}_I^{3,D}$ was used to obtain a winning strategy for player I, whereas $\text{ORD}_{II,E}^{3,T^{1,D}}$ (not $\text{ORD}_{II}^{3,E}$) was used to obtain a winning strategy for player II.

We next verify that our proof was not dependent on our choice of players with open payoff sets and the non-helping conditions for the three players. Suppose X, Y, Z are players I, II, III in an arbitrary order. We wish to verify that our proof of Theorem 3.19 gives determinacy of the three-player biased open game in which X, Y and Z respectively take the roles that I, II and III had in Theorem 3.19.

Definition 5.1. Definition of the canonical three-player biased open games.

(i) Let D and E be perpendicular sets of positions, let $\{X, Y, Z\} = \{I, II, III\}$, and

let $T^{X,D} = \{p \mid \forall q \subseteq p \text{ ORD}_X^{3,D}(q) \uparrow\}$. We define $G_{(X,Y,Z)}(D, E)$ to be the

three-player biased open game in which players X and Y respectively have

$O(D)$ and $O(E)$ as payoff sets and in which G satisfies: $Y \xrightarrow[D]{\text{help}} X$,

$X \xrightarrow[E]{T^{X,D} \text{ help}} Y$, and $Z \xrightarrow[D]{\text{help}} X$.

(ii) A three-player game G is a *canonical three-player biased open game* if G is

$G_{(X,Y,Z)}(D, E)$ for some permutation X, Y, Z of players I, II, III and for some

any sets D and E of positions such that $D \perp E$.

(iii) Let $(Y, E) \xrightarrow[D]{\text{help}} (X, D)$ abbreviate $Y \xrightarrow[D]{\text{help}} X$ and $X \xrightarrow[E]{T^{X,D} \text{ help}} Y$.

(Recall the definition of $\text{ORD}_{Y,E}^{T^{X,D}}$ from Definition 3.8. We drop D and E if these are clear from the context.)

Notice $(Y, E) \xrightarrow[D]{\text{help}} (X, D)$ is not equivalent to $(X, D) \xrightarrow[E]{\text{help}} (Y, E)$, which

abbreviates $X \xrightarrow[E]{\text{help}} Y$ and $Y \xrightarrow[D]{T^{Y,E} \text{ help}} X$.

At the end of this section, in Theorem 5.4, we prove that all canonical three-player biased open games are determined, i.e. we show $G_{(X_1, X_2, X_3)}(D_1, D_2)$ is determined for any sets D_1 and D_2 of positions such that $D_1 \perp D_2$ and for any permutation X_1, X_2, X_3 of I, II, III . We first prove a few special cases of this in Theorem 5.2 and 5.3 as a warm-up. The reader can skip to Theorem 5.4 and its proof without any loss of generality.

Our initial case is a proof in which players I and III have open payoff sets.

Theorem 5.2. $G_{(I, III, II)}(D, E)$ is determined for any pair of sets D and E of positions such that $D \perp E$. That is, we have the determinacy of any three-player biased open game G with players I and III having open payoff sets $O(D)$ and $O(E)$, and in which G satisfies $\text{III} \xrightarrow[\text{D}]{\text{help}} \text{I}$, $\text{I} \xrightarrow[\text{E}]{\text{T}^{\text{LD}} \text{ help}} \text{III}$ and $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$.

Proof. Suppose $D \perp E$. Let us consider a three-player game G in which players I and III respectively have open payoff sets $O(D)$ and $O(E)$:

$$G: \begin{array}{ccccccc} & \text{I} & a_0 & & a_3 & & a_{3n} \\ & \text{II} & & a_1 & & a_4 & \text{L} & & a_{3n+1} & & \text{L} & . \\ & \text{III} & & & a_2 & & & a_5 & & & & a_{3n+2} \end{array}$$

We shall eventually show that G is determined if G satisfies $(\text{III}, E) \xleftrightarrow[\text{D}]{\text{help}} (\text{I}, D)$ and $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$. In particular, the canonical three-player biased open game

$G_{(I, III, II)}(D, E)$ is determined. (Recall $\text{ORD}_I^{3,D}, \text{T}^{\text{LD}} = \{ \frac{r}{p} \mid \forall q \subseteq \frac{r}{p} \text{ORD}_I^{3,D}(q) \uparrow \}$, and $\text{ORD}_{\text{III},E}^{3,\text{T}^{\text{LD}}}$ from Definitions 3.9 and 5.1.)

First let's summarize which non-helping conditions shall be used to construct winning strategies for which players. When $\langle \rangle \in \text{ORD}_I^{3,D}$, no non-helping conditions shall be required to define a winning strategy for player I. When $\langle \rangle \in \text{ORD}_{\text{III},E}^{3,\text{T}^{\text{LD}}}$, we shall use $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$ to define a winning strategy for player III. When $\langle \rangle \in \text{T}^{\text{LD}} \setminus \text{ORD}_{\text{III},E}^{3,\text{T}^{\text{LD}}}$,

we shall use $(\text{III}, E) \xrightarrow{\text{help}} (I, D)$ to define a winning strategy for player II.

Claim 4. G is determined if G satisfies $\text{III}, (E) \xrightarrow{\text{help}} I, (D)$ and $\text{II} \xrightarrow{\text{help}_D} I$.

This follows from Claims 1, 2, and 3 below.

Claim 1. If $\langle \rangle \in \text{ORD}_I^{3,D}$, then player I has a winning strategy for G .

Suppose $\langle \rangle \in \text{ORD}_I^{3,D}$. By the definition of $\text{ORD}_I^{3,D}$, $\exists a_0$ such that

$\text{ORD}_I^{3,D}(a_0) < \text{ORD}_I^{3,D}(\langle \rangle)$ if $\text{ORD}_I^{3,D}(\langle \rangle) \neq 0$. In this case, player I's strategy is to play

a_0 such that $\text{ORD}_I^{3,D}(a_0) \downarrow$ and is $< \text{ORD}_I^{3,D}(\langle \rangle)$. By the definition of $\text{ORD}_I^{3,D}$,

$\forall a_1 [\text{ORD}_I^{3,D}(a_0, a_1) \leq \text{ORD}_I^{3,D}(a_0)]$. Therefore, once player II plays a_1 , we have

$\text{ORD}_I^{3,D}(a_0, a_1) \downarrow$ and is $\leq \text{ORD}_I^{3,D}(a_0)$. By the definition of $\text{ORD}_I^{3,D}$,

$\forall a_2 [\text{ORD}_I^{3,D}(a_0, a_1, a_2) \leq \text{ORD}_I^{3,D}(a_0, a_1)]$. So, once player III plays a_2 , we have

$\text{ORD}_I^{3,D}(a_0, a_1, a_2) \downarrow$ and is $\leq \text{ORD}_I^{3,D}(a_0, a_1)$. Suppose play continues through each

inning in this manner with player I playing according to his strategy to strictly decrease

the value of $\text{ORD}_I^{3,D}$. Eventually a position is reached with $\text{ORD}_I^{3,D}$ -value of zero since a

decreasing sequence of ordinals is finite; then player I wins the game. , (Claim 1)

Claim 2. If $\langle \rangle \in \text{ORD}_{\text{III}, E}^{3, T^1, D}$ and G satisfies $\text{II} \xrightarrow{\text{help}_D} I$, then player III has a winning strategy for G .

Suppose $\langle \rangle \in \text{ORD}_{\text{III}, E}^{3, T^1, D}$ and G satisfies $\text{II} \xrightarrow{\text{help}_D} I$. Since $\langle \rangle \in T^{1,D}$, by the

definition of $\text{ORD}_I^{3,D}$, $\forall a_0 \left[(a_0) \in T^{1,D} \right]$ (i.e. $(a_0) \notin \text{ORD}_I^{3,D}$). By the definition of

$\text{ORD}_{III,E}^{3,T^{1,D}}$, $\forall a_0 \in T^{1,D} \left[\text{ORD}_{III,E}^{3,T^{1,D}}(a_0) \leq \text{ORD}_{III,E}^{3,T^{1,D}}(\langle \rangle) \right]$. Therefore, once player I plays a_0 ,

we have $\text{ORD}_{III,E}^{3,T^{1,D}}(a_0) \downarrow$ and is $\leq \text{ORD}_{III,E}^{3,T^{1,D}}(\langle \rangle)$. We shall show $\forall^L a_1 \left[a_1 \in T_{(a_0)}^{1,D} \right]^1$

follows from $(a_0) \in T^{1,D}$ and $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$ by what we shall call the Sup/NH (Sup/Non-Helping) Argument.

Sup/NH Argument Since $(a_0) \in T^{1,D}$, by the definition of $\text{ORD}_I^{3,D}$, $(a_0, a_1) \notin \text{ORD}_I^{3,D}$ for some move $a_1 \in T_{(a_0)}^{1,D}$ by player II so that it is possible for player II not to help player I.

Therefore, since $\text{II} \xrightarrow[\text{D}]{\text{help}} \text{I}$, we have $(a_0, a_1) \notin \text{ORD}_I^{3,D}$ for any legal

move a_1 , i.e. $\forall^L a_1 \left[a_1 \in T_{(a_0)}^{1,D} \right]$. , (Sup/NH Argument)

By the definition of $\text{ORD}_{III,E}^{3,T^{1,D}}$, $\forall a_1 \in T_{(a_0)}^{1,D} \left[\text{ORD}_{III,E}^{3,T^{1,D}}(a_0, a_1) \leq \text{ORD}_{III,E}^{3,T^{1,D}}(a_0) \right]$. Therefore,

since $\forall^L a_1 \left[a_1 \in T_{(a_0)}^{1,D} \right]$, once player II plays a legal move a_1 , we have $\text{ORD}_{III,E}^{3,T^{1,D}}(a_0, a_1) \downarrow$

and is $\leq \text{ORD}_{III,E}^{3,T^{1,D}}(a_0)$. By the definition of $\text{ORD}_{III,E}^{3,T^{1,D}}$,

$\exists a_2 \in T_{(a_0, a_1)}^{1,D} \left[\text{ORD}_{III,E}^{3,T^{1,D}}(a_0, a_1, a_2) < \text{ORD}_{III,E}^{3,T^{1,D}}(a_0, a_1) \right]$ unless $\text{ORD}_{III,E}^{3,T^{1,D}}(a_0, a_1) = 0$.

Player III's strategy is to play such an a_2 whenever $\text{ORD}_{III,E}^{3,T^{1,D}}(a_0, a_1) > 0$. Suppose play continues through each inning in this manner with player III playing according to his

strategy to strictly decrease the value of $\text{ORD}_{III,E}^{3,T^{1,D}}$. Eventually a position is reached with

$\text{ORD}_{III,E}^{3,T^{1,D}}$ - value of zero since a decreasing sequence of ordinals is finite; then player III

¹ $\forall^L a_1$ abbreviates "for all legal moves a_1 ".

wins the game.

, (Claim 2)

Claim 3. If $\langle \rangle \in T^{1,D} \setminus \text{ORD}_{\text{III},E}^{3,T^{1,D}}$ (i.e. $\langle \rangle \notin \text{ORD}_I^{3,D}$ and $\langle \rangle \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}$) and G satisfies

$(\text{III}, E) \xrightarrow{\text{help}} (\text{I}, D)$, then player II has a winning strategy.

Suppose $\langle \rangle \in T^{1,D} \setminus \text{ORD}_{\text{III},E}^{3,T^{1,D}}$ and G satisfies $(\text{III}, E) \xrightarrow{\text{help}} (\text{I}, D)$. Since

$\langle \rangle \notin \text{ORD}_I^{3,D}$, by the definition of $\text{ORD}_I^{3,D} \forall a_0 [(a_0) \in T^{1,D}]$. Since $\langle \rangle \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}$ and

$\text{I} \xrightarrow{\text{help}} \text{III}$, by the Sup/NH Argument, we have $\forall^L a_0 [a_0 \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}]$.² Therefore,

since $\forall a_0 [(a_0) \in T^{1,D}]$, $\forall^L a_0 [(a_0) \in T^{1,D} \setminus \text{ORD}_{\text{III},E}^{3,T^{1,D}}]$. Since $(a_0) \in T^{1,D} \setminus \text{ORD}_{\text{III},E}^{3,T^{1,D}}$, by the

usual sup-argument, there exists $a_1 \in T_{(a_0)}^{1,D}$ such that $(a_0, a_1) \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}$; it is player II's

strategy to play such an a_1 (so that $(a_0, a_1) \in T^{1,D} \setminus \text{ORD}_{\text{III},E}^{3,T^{1,D}}$). Since $(a_0, a_1) \in T^{1,D}$ and

$\text{III} \xrightarrow{\text{help}} \text{I}$, by the Sup/NH Argument, we have $\forall^L a_2 [(a_0, a_1, a_2) \notin \text{ORD}_I^{3,D}]$, i.e.

$\forall^L a_2 [a_2 \in T_{(a_0, a_1)}^{1,D}]$.³ Since $(a_0, a_1) \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}$, by the definition of $\text{ORD}_{\text{III},E}^{3,T^{1,D}}$,

$(a_0, a_1, a_2) \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}$. Therefore, once player III plays a legal move a_2 , we have

$(a_0, a_1, a_2) \in T^{1,D} \setminus \text{ORD}_{\text{III},E}^{3,T^{1,D}}$ for any move $a_2 \in T_{(a_0, a_1)}^{1,D}$. Suppose play continues through

² Since $\langle \rangle \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}$, by the usual sup-argument for some move $a_0 \in T^{1,D}$, $a_0 \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}$. Therefore it

is possible for player I not to help player III. Since $\text{I} \xrightarrow{\text{help}} \text{III}$, for any legal move a_0 by player I,

we have $a_0 \notin \text{ORD}_{\text{III},E}^{3,T^{1,D}}$.

³ Since $(a_0, a_1) \notin \text{ORD}_I^{3,D}$, by the usual sup-argument for some $a_2 \in T_{(a_0, a_1)}^{1,D}$, $(a_0, a_1, a_2) \notin \text{ORD}_I^{3,D}$.

Therefore it is possible for player III not to help player I. Since $\text{III} \xrightarrow{\text{help}} \text{I}$, for any legal move a_2 by

player III, we have $(a_0, a_1, a_2) \notin \text{ORD}_I^{3,D}$.

each inning in the above manner with player II playing according to his strategy to keep the play out of $\text{ORD}_I^{3,D}$ and $\text{ORD}_{III,E}^{3,T^{1,D}}$. Player I will not win since the $\text{ORD}_I^{3,D}$ -value diverges at every position (and will never reach zero). Player III will not win since the $\text{ORD}_{III,E}^{3,T^{1,D}}$ -value diverges at every position (and will never reach zero). Therefore player II will win if he plays according to the above strategy. , (Claim 3)

Consequently, by Claims 1, 2, and 3, determined is the three-player biased open game

$G_{(I, III, II)}(D, E)$ which satisfies $(III, E) \xrightarrow{\text{help}} (I, D)$ and $II \xrightarrow{\text{help}} I$. , (Theorem 5.2)

Our next proof will study the determinacy of a three-player biased open game in which players II and III have open payoff sets.

Theorem 5.3. $G_{(II, III, I)}(D, E)$ is determined for any pair of sets D and E of positions such that $D \perp E$. That is, we have the determinacy of any three-player biased open game G with players II and III having open payoff sets $O(D)$ and $O(E)$ and in which G satisfies $III \xrightarrow{\text{help}} II$, $II \xrightarrow{T^{II,D}, \text{help}} III$, and $I \xrightarrow{\text{help}} II$.

Proof. The proof is essentially the same as that of Theorem 5.2 except here players I and II switch their roles from that in Theorem 5.2. Assume $D \perp E$. Let us consider a three-player game G in which players II and III respectively have open payoff sets $O(D)$ and

$O(E)$:

$$\begin{array}{cccccccc}
 & \text{I} & a_0 & & a_3 & & a_{3n} & \\
 G: & \text{II} & & a_1 & & a_4 & \text{L} & a_{3n+1} & \text{L} & . \\
 & \text{III} & & & a_2 & & & a_5 & & a_{3n+2}
 \end{array}$$

We shall eventually show that G is determined if G satisfies $(\text{III}, E) \xleftrightarrow{\text{help}} (\text{II}, D)$

and $\text{I} \xrightarrow[\text{D}]{\text{help}} \text{II}$. In particular, the canonical three-player biased open game

$G_{(\text{II}, \text{III}, \text{I})}(D, E)$ is determined. (Recall $\text{ORD}_{\text{II}}^{3, \text{D}}$, $\text{T}^{\text{II}, \text{D}} = \{ \overset{\text{r}}{p} \mid \forall \overset{\text{r}}{q} \subseteq \overset{\text{r}}{p} \text{ORD}_{\text{II}}^{3, \text{D}}(\overset{\text{r}}{q}) \uparrow \}$ and

$\text{ORD}_{\text{III}, \text{E}}^{3, \text{T}^{\text{II}, \text{D}}}$ from Definitions 3.9 and 5.1.)

First let's summarize which non-helping conditions are used to construct winning strategies for which players. When $\langle \rangle \in \text{ORD}_{\text{II}}^{3, \text{D}}$, no non-helping conditions shall be

required to define a winning strategy for player II. When $\langle \rangle \in \text{ORD}_{\text{III}, \text{E}}^{3, \text{T}^{\text{II}, \text{D}}}$, we shall use

$\text{I} \xrightarrow[\text{D}]{\text{help}} \text{II}$ to define a winning strategy for player III. When $\langle \rangle \in \text{T}^{\text{II}, \text{D}} \setminus \text{ORD}_{\text{III}, \text{E}}^{3, \text{T}^{\text{II}, \text{D}}}$, we

shall use $(\text{III}, E) \xleftrightarrow{\text{help}} (\text{II}, D)$ to define a winning strategy for player I.

Claim.4. G is determined if G satisfies $(\text{III}, E) \xleftrightarrow{\text{help}} (\text{II}, D)$ and $\text{I} \xrightarrow[\text{D}]{\text{help}} \text{II}$.

This follows from Claims 1, 2, and 3, below.

Claim 1. If $\langle \rangle \in \text{ORD}_{\text{II}}^{3, \text{D}}$, then player II has a winning strategy for G .

Assume $\langle \rangle \in \text{ORD}_{\text{II}}^{3, \text{D}}$. The strategy for player II is to play a move that strictly

decreases the value of $\text{ORD}_{\text{II}}^{3, \text{D}}$ until we reach a position with $\text{ORD}_{\text{II}}^{3, \text{D}}$ -value zero. Due to

the definition of $\text{ORD}_{\text{II}}^{3,\text{D}}$, no move by player I or III can take a position outside $\text{ORD}_{\text{II}}^{3,\text{D}}$ or to a position with a higher $\text{ORD}_{\text{II}}^{3,\text{D}}$ -value. Eventually any play according to player II's strategy reaches a position with $\text{ORD}_{\text{II}}^{3,\text{D}}$ -value of zero since a decreasing sequence of ordinals is finite. Consequently, any such play is a win for player II. , (Claim 1)

Claim 2. If $\langle \rangle \in \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$ and G satisfies $\text{I} \xrightarrow[\text{D}]{\text{help}} \text{II}$, then player III has a winning strategy for the game G .

Assume $\langle \rangle \in \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$ and G satisfies $\text{I} \xrightarrow[\text{D}]{\text{help}} \text{II}$. Recall that here player I and II switch the roles they had in Theorem 5.2. Otherwise this case is similar to Claim 2 of Theorem 5.2. Since $\langle \rangle \in \text{T}^{\text{II,D}}$ and $\text{I} \xrightarrow[\text{D}]{\text{help}} \text{II}$, by the Sup/NH Argument,

$\forall^L a_0 [(a_0) \in \text{T}^{\text{II,D}}]$. Since $\langle \rangle \in \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$, $\forall (a_0) \in \text{T}^{\text{II,D}}$, $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0) \downarrow$ and is $\leq \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(\langle \rangle)$ by the definition of $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$. Therefore $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0) \downarrow$ and is $\leq \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(\langle \rangle)$ for any legal move a_0 . Since $(a_0) \in \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$, $(a_0) \in \text{T}^{\text{II,D}}$ so that by the definition of $\text{ORD}_{\text{II}}^{3,\text{D}}$, $(a_0, a_1) \in \text{T}^{\text{II,D}}$ for any move a_1 . Since $\langle \rangle \in \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$,

$\forall a_1 \in \text{T}_{(a_0)}^{\text{II,D}}$ $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0, a_1) \downarrow$ and is $\leq \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0)$. Therefore $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0, a_1) \downarrow$ and is $\leq \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0)$ for any move a_1 . Since $(a_0, a_1) \in \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$, by the definition of

$\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$ there is an $a_2 \in \text{T}_{(a_0, a_1)}^{\text{II,D}}$ such that $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0, a_1, a_2)$ has a strictly lower

$\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$ -value than $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0, a_1)$ if $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}(a_0, a_1) > 0$. In this case, player III's

strategy is to play such an a_2 . Eventually any play according to player III's strategy

reaches a position with $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$ -value zero since a decreasing sequence of ordinals is finite. Consequently, any such play is a win for player III. , (Claim 2)

Claim 3. If $\langle \rangle \in \text{T}^{\text{III,E}} =_{\text{df}} \text{T}^{\text{II,D}} \setminus \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$ (i.e. $\langle \rangle \notin \text{ORD}_{\text{II}}^{3,\text{D}}$ and $\langle \rangle \notin \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$) and G

satisfies $(\text{III}, E) \xrightarrow{\text{help}} (\text{II}, D)$, then player I has a winning strategy for the game G .

Assume $\langle \rangle \in \text{T}^{\text{II,D}} \setminus \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$ and satisfies $(\text{III}, E) \xrightarrow{\text{help}} (\text{II}, D)$. This claim is similar to Claim 3 in Theorem 5.2. Since $\langle \rangle \notin \text{ORD}_{\text{II}}^{3,\text{D}}$, $\langle \rangle \notin \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$, and since $(\text{III}, E) \xrightarrow{\text{help}} (\text{II}, D)$, any legal move by players II and III will keep the play out of each other's ordinal, by the Sup/NH Argument. Since $\langle \rangle \notin \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$, $\exists a_0 [(a_0) \in \text{T}^{\text{II,D}}]$ such that $(a_0) \notin \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$; player I's strategy is to player such an $(a_0) \in \text{T}^{\text{III,E}}$, i.e $(a_0) \in \text{T}^{\text{II,D}} \setminus \text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$. In general, player I's strategy is to play to stay in $\text{T}^{\text{III,E}}$ and therefore stay out of $\text{ORD}_{\text{II}}^{3,\text{D}}$ and $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$. This strategy results in a win for player I since any position according to this strategy does not have an ordinal-value of zero for $\text{ORD}_{\text{II}}^{3,\text{D}}$ and $\text{ORD}_{\text{III,E}}^{3,\text{T}^{\text{II,D}}}$. , (Claim 3)

Consequently, by Claims 1, 2, and 3, determined is the game $G_{(\text{II}, \text{III}, \text{D})}(D, E)$ which satisfies $(\text{III}, E) \xrightarrow{\text{help}} (\text{II}, D)$ and $\text{I} \xrightarrow{\text{help}} \text{II}$. , (Theorem 5.3)

The next theorem is a generalized proof of Theorems 3.20, 5.2 and 5.3 that all

canonical three-player biased open games are determined.

Theorem 5.4. All canonical three-player biased open games are determined, i.e. for $\{X_1, X_2, X_3\} = \{I, II, III\}$ and any pair of sets D_1 and D_2 of positions such that $D_1 \perp D_2$,

$G_{(X_1, X_2, X_3)}(D_1, D_2)$ is determined. That is, we have the determinacy of any three-player biased open game G with players X_1 and X_2 having open payoff sets $O(D_1)$ and

$O(D_2)$ and in which G satisfies: $X_2 \xrightarrow[\cancel{D_1}]{\text{help}} X_1$, $X_1 \xrightarrow[\cancel{D_2}]{T^{X_1, D_1} \text{ help}} X_2$, and

$X_3 \xrightarrow[\cancel{D_1}]{\text{help}} X_1$.

Proof. Let D_1 and D_2 be sets of positions such that $D_1 \perp D_2$. Let

$\{X_1, X_2, X_3\} = \{I, II, III\}$. Let G be a three-player game in which players X_1 and X_2 respectively have open payoff sets $O(D_1)$ and $O(D_2)$. We will eventually show that G

is determined if G satisfies $(X_1, D_1) \xrightarrow[\cancel{D_2}]{\text{help}} (X_2, D_2)$ and $X_3 \xrightarrow[\cancel{D_1}]{\text{help}} X_1$. That is, the

canonical three-player game $G_{(X_1, X_2, X_3)}(D_1, D_2)$ is determined.

Let T^{X_0} be the entire game tree for the game. Inductively define $T^{X_{i-1}}$ as follows:

- (i) $T^{X_i, D_i} = T^{X_{i-1}, D_{i-1}} \setminus \text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}$ for $i \leq 2$.

Recall $\text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}$ is defined in Definition 3.5? so that:

- (ii) $\text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}$ denotes ordinals of positions with respect to D_i and for player X_i

in the game tree $T^{X_{i-1}, D_{i-1}}$. In particular $\text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}(\mathbf{r}) \downarrow \Rightarrow \mathbf{r} \in T^{X_{i-1}, D_{i-1}}$.

(iii) $\text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}(\bar{p}) = 0$ iff $\bar{p} \in T^{X_{i-1}, D_{i-1}}$ and $\exists q \in D_i$ ($\bar{p} \supseteq \bar{q}$).

It easily follows that:

(iv) A play \bar{y} is a win for player X_i iff $\exists k \bar{y}(k) \in D_i$

iff $\exists n \geq k \text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}(\bar{y}(n)) = 0$.

Let's first summarize which non-helping conditions are used to construct winning strategies for which players. When $\langle \rangle \in \text{ORD}_{X_1}^{3, D_1}$, no non-helping conditions shall be

required to define a winning strategy for player X_1 . When $\langle \rangle \in \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$, we shall use

$X_3 \xrightarrow[D_1]{\text{help}} X_1$ to define a winning strategy for player X_2 . When $\langle \rangle \in T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$,

we shall use the non-helping condition $(X_1, D_1) \xrightarrow{\text{help}} (X_2, D_2)$ to define a winning strategy for player X_3 .

We now generalize an often repeated Sup/Non-Helping Argument, introduced in Theorem 5.2:

Sup/NH Lemma.⁴ For a three player biased open game and for $i = 1, 2$, we have: If

$\bar{p} \in T^{X_i, D_i}$, $\forall^L m \left[\bar{p} \right](m) \in T^{X_{i-1}, D_{i-1}}$, U is a player different than player X_i such that

$U \xrightarrow[D_i]{T^{X_{i-1}, D_{i-1}}, \text{help}} X_i$, and if it is player U 's turn to move at position \bar{p} , then

$\forall^L m \left[\bar{p} \right](m) \in T^{X_i, D_i}$.

⁴ We will use the Sup/Non-Helping Lemma again in Chapter 6.

Proof. Suppose $\overset{\mathbf{r}}{p} \in T^{X_i, D_i}$, $\forall^L m \left[\overset{\mathbf{r}}{p} \right) (m) \in T^{X_{i-1}, D_{i-1}} \right]$, $U \neq X_i$, $U \xrightarrow[\cancel{D_i}]{\cancel{T^{X_{i-1}, D_{i-1}} \text{ help}}} X_i$, and it is not player X_i 's turn to move. Show $\overset{\mathbf{r}}{p} \right) (m) \in T^{X_{i-1}, D_{i-1}}$ and $\overset{\mathbf{r}}{p} \right) (m) \notin \text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}$ for any legal move m .

Since $\overset{\mathbf{r}}{p} \in T^{X_{i-1}, D_{i-1}}$, $\overset{\mathbf{r}}{p} \notin \text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}$, and since $U \neq X_i$, by the sup-argument $\exists m \in T_{\overset{\mathbf{r}}{p}}^{X_{i-1}, D_{i-1}} \left[\overset{\mathbf{r}}{p} \right) (m) \notin \text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}} \right]$. Therefore, it is possible for player U not to help player X_i . Since $U \xrightarrow[\cancel{D_i}]{\cancel{T^{X_{i-1}, D_{i-1}} \text{ help}}} X_i$, for any legal move m by player U , $\overset{\mathbf{r}}{p} \right) (m) \notin \text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}}$. Since $\forall^L m \left[\overset{\mathbf{r}}{p} \right) (m) \in T^{X_{i-1}, D_{i-1}} \right]$ (by the hypothesis to the lemma), $\forall^L m \left[\overset{\mathbf{r}}{p} \right) (m) \in T^{X_i, D_i} = T^{X_{i-1}, D_{i-1}} \setminus \text{ORD}_{X_i, D_i}^{3, T^{X_{i-1}, D_{i-1}}} \right]$. , (Sup/NH Lemma)

Claim 4. G is determined if G satisfies $(X_1, D_1) \xleftrightarrow[\cancel{\text{help}}]{\cancel{\text{help}}} (X_2, D_2)$ and $X_3 \xrightarrow[\cancel{\text{help}}]{\cancel{\text{help}}} (X_1, D_1)$, that is the canonical three-player biased open game $G_{(X_1, X_2, X_3)}(D_1, D_2)$ is determined.

This claim follows from Claims 1, 2, and 3, below.

Claim 1. If $\langle \rangle \in \text{ORD}_{X_1}^{3, D_1}$, then player I has a winning strategy for G .

Assume $\langle \rangle \in \text{ORD}_{X_1}^{3, D_1}$. The strategy for player X_1 is to play a move that strictly decreases the value of $\text{ORD}_{X_1}^{3, D_1}$ until we reach a position with $\text{ORD}_{X_1}^{3, D_1}$ -value of zero. No move by player X_2 or player X_3 can take a position in $\text{ORD}_{X_1}^{3, D_1}$ to a position outside

$\text{ORD}_{X_1}^{3,D_1}$ or with higher $\text{ORD}_{X_1}^{3,D_1}$ -value. Therefore any position completing a later inning will have a strictly lower $\text{ORD}_{X_1}^{3,D_1}$ -value than positions from earlier innings with nonzero $\text{ORD}_{X_1}^{3,D_1}$ -value. Therefore, since a strictly decreasing sequence of ordinals is finite, eventually a position is reached with $\text{ORD}_{X_1}^{3,D_1}$ -value of zero, so that player X_1 wins when he plays according to this strategy. , (Claim 1)

Claim 2. If $\langle \rangle \in \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ (so that $\langle \rangle \notin \text{ORD}_{X_1}^{3, D_1}$) and G satisfies $X_3 \xrightarrow[\cancel{D_1}]{\text{help}} X_1$, then player X_2 has a winning strategy for the game G .

Assume $\langle \rangle \in \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ and G satisfies $X_3 \xrightarrow[\cancel{D_1}]{\text{help}} X_1$. We will describe a strategy for player X_2 such that positions according to player X_2 's strategy are in $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ (and therefore in T^{X_1, D_1}) Player X_2 's strategy will be to play a move that strictly decreases the value of $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ until we reach a position with $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ -value of zero. Since $\langle \rangle \in \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$, $\langle \rangle \in T^{X_1, D_1}$.

Consider an arbitrary position $\overset{\mathbf{r}}{p}$ such that $\overset{\mathbf{r}}{p} \in \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ (and therefore $\overset{\mathbf{r}}{p} \in T^{X_1, D_1}$).

We shall first show that if it is player X_1 or X_3 's turn, then for every legal move m :

$$\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1}, \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p}(m)) \downarrow, \text{and } \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p}(m)) \leq \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p}).$$

If it is player X_1 's turn to move, $\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1}$ for any move m by the definition of $\text{ORD}_{X_1}^{3, D_1}$. If it is player X_3 's turn to move, then by the Sup/NH Lemma

$\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1} \right]$, since : $\overset{\mathbf{r}}{p} \in T^{X_1, D_1}$, $\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_0} \right]$, $X_3 \xrightarrow[\cancel{D_1}]{\text{help}} X_1$ and it is

not player X_1 's turn to move. Thus in either case, $\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1}$, and since

$\overset{\mathbf{r}}{p} \in \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$, $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p}(m)) \downarrow$ and $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p}(m)) \leq \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p})$ by the

definition of $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$.

Now suppose it is player X_2 's turn to move. In this case, we describe player X_2 's

strategy. Recall $\overset{\mathbf{r}}{p} \in \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$. If $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p}) > 0$, by the definition of $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$

there exists a move $m \in T_{\overset{\mathbf{r}}{p}}^{X_1, D_1}$ such that $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p}(m)) \downarrow$ and

$\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p}(m)) < \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}(\overset{\mathbf{r}}{p})$; it is player X_2 's strategy to play such an m .

In summary, any legal move by players X_1 and X_3 takes a position in $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ to a

position in $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ with no larger $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ -value and player X_2 's strategy results in

his making moves which strictly decreases the $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ -value until we reach a position

with $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ -value of zero. Since a decreasing sequence of ordinals is finite,

eventually a position is reached with $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ -value of zero, so that player X_2 wins

when he plays according to his strategy. , (Claim 2)

Claim 3. If $\langle \rangle \in T^{X_2, D_2} =_{\text{df}} T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ (i.e. $\langle \rangle \notin \text{ORD}_{X_1}^{3, D_1}$ and $\langle \rangle \notin \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$)

and G satisfies $(X_1, D_1) \xrightarrow[\cancel{D_1}]{\text{help}} (X_2, D_2)$, then player X_3 has a winning strategy for G .

Assume $\langle \rangle \in T^{X_2, D_2} = T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ and G satisfies $(X_1, D_1) \xrightarrow[\cancel{D_1}]{\text{help}} (X_2, D_2)$.

We will describe a strategy for player X_3 such that positions according to player X_3 's strategy are in $T^{X_2, D_2} = T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$; then we shall show that the strategy must be a winning strategy for player X_3 . We shall first show that no move by player X_1 or player X_2 can take a position in $T^{X_2, D_2} = T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ to a position outside $T^{X_2, D_2} = T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$. Then we describe the strategy for player X_3 .

Consider an arbitrary position $\overset{\mathbf{r}}{p}$ such that $\overset{\mathbf{r}}{p} \in T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$. First suppose it is player X_1 's turn to move. By the definition of $\text{ORD}_{X_1}^{3, D_1}$, $\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1}$ for any move m by player X_1 . Then $\forall^L m [\overset{\mathbf{r}}{p}(m) \in T^{X_2, D_2}]$ by the Sup/NH Lemma, since: $\overset{\mathbf{r}}{p} \in T^{X_1, D_1}$, $\forall^L m [\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1}]$, $X_1 \xrightarrow[\text{D}_2]{\text{help}} X_2$, and it is not player X_2 's turn to move.

Therefore, since $\overset{\mathbf{r}}{p} \notin \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$, for all legal moves m by player X_1 ,

$$\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}.$$

Now suppose it is player X_2 's turn to move. Then $\forall^L m [\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1}]$ by the Sup/NH Lemma since: $\overset{\mathbf{r}}{p} \in T^{X_1, D_1}$, $\forall^L m [\overset{\mathbf{r}}{p}(m) \in T^{X_0}]$, $X_2 \xrightarrow[\text{D}_1]{\text{help}} X_1$, and it is not player X_1 's turn to move. Since $\overset{\mathbf{r}}{p}(m) \in T^{X_1, D_1}$ and $\overset{\mathbf{r}}{p} \notin \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$, by the definition of $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$, $\forall m \in T_{\overset{\mathbf{r}}{p}}^{X_1, D_1} [\overset{\mathbf{r}}{p}(m) \notin \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}]$. Therefore, for all legal moves m by player X_2 , $\overset{\mathbf{r}}{p}(m) \in T^{X_2, D_2} = T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$.

In summary, any legal move by player X_1 or X_2 takes a position in T^{X_2, D_2} to a position in T^{X_2, D_2} .

Now suppose it is player X_3 's turn to move.⁵ Since $\overset{\Gamma}{p} \in T^{X_1, D_1}$ and $\overset{\Gamma}{p} \notin \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$, by the usual sup-argument $\exists m \in T_{\overset{\Gamma}{p}}^{X_1, D_1}$ $\overset{\Gamma}{p}'(m) \notin \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$.⁶ The strategy for player X_3 is to play such an m in which $\overset{\Gamma}{p}'(m) \in T^{X_2, D_2} = T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$.

Consequently we have shown that:

$$\overset{\Gamma}{p} \in T^{X_2, D_2} \Rightarrow \overset{\Gamma}{p}'(m) \in T^{X_2, D_2} \quad (\mathfrak{S})$$

when m is a legal move by players X_1 or X_2 or when m is a move by player X_3 and $\overset{\Gamma}{p}'(m)$ is according to player X_3 's strategy.

Suppose $\overset{\Gamma}{y}$ is a play according to player X_3 's strategy. Since $\langle \rangle \in T^{X_2, D_2}$, by (\mathfrak{S}) , $\bar{y}(n) \in T^{X_2, D_2} = T^{X_1, D_1} \setminus \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ for every n . Therefore $\bar{y}(n)$ has neither $\text{ORD}_{X_1}^{3, D_1}$ -value of zero nor $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ -value of zero so that neither player X_1 nor X_2 wins. Thus $\overset{\Gamma}{y}$ is a win for player X_3 . , (Claim 3)

Thus, by Claims 1, 2, and 3, the canonical three-player biased open game

$G_{(X_1, X_2, X_3)}(D_1, D_2)$ is determined. , (Theorem 5.4)

As we noted after Theorem 3.19, the same conclusion can be formed after Theorem 5.4. Players can be restricted to only make moves that do not stray outside the existing

⁵ Since $\overset{\Gamma}{p} \in T^{X_1, D_1}$ by the usual sup-argument there exists a move m such that $\overset{\Gamma}{p}'(m) \in T^{X_1, D_1}$. Therefore, since $X_3 \xrightarrow[\text{D}_1]{\text{help}} X_1$, $\overset{\Gamma}{p}'(m) \in T^{X_1, D_1}$ for any legal move by player X_3 .

⁶ That we can find one m such that both $m \in T_{\overset{\Gamma}{p}}^{X_1, D_1}$ and $\overset{\Gamma}{p}'(m) \notin \text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$ is the reason we used $\text{ORD}_{X_2}^{3, D_2}$ instead of $\text{ORD}_{X_2, D_2}^{3, T^{X_1, D_1}}$.

ordinal values of a position in a canonical three-player game:

Corollary 5.5. Let D_1 and D_2 be perpendicular sets of positions and

$\{X_1, X_2, X_3\} = \{I, II, III\}$ represent the three players in any order. Determined is any

infinite three-player game of perfect information in which:

- (i) at most one player has a payoff set that is not open,
- (ii) at every position, there is a move m such that at the resulting position, no player other than possibly the player making the move m has a winning strategy, and
- (iii) each player is required to make such a move m .

CHAPTER 6

DETERMINACY OF MULTIPLAYER BIASED GAMES

In Chapter 5 we showed that with certain restrictions on a player “not helping” his opponent we had determinacy of three-player biased open games. Theorem 5.4 gave us a general proof to obtain determinacy of three-player biased open games. In Chapter 6 we generalize the proof (of Theorem 5.4) to obtain determinacy of four-player biased open games, and then further generalize the proof to obtain determinacy of multiplayer biased open games.

In Definition 6.2 below, we generalize the definition of the three-player game

$G_{(X_1, X_2, X_3)}(D_1, D_2)$ (from Definition 5.1(ii)) to the four-player game

$G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$ and in Definition 6.3 to (n+1)-player games

$G_{(X_1, X_2, \dots, X_{n+1})}(D_1, D_2, \dots, D_n)$.

$G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$ we define to be the four-player biased open game in

which:

- (i) Player X_i has open payoff set $O(D_i)$ for $1 \leq i \leq 3$.
- (ii) Player X_i cannot help player Y for every X and for every $Y \notin \{X, X_4\}$ such

that $(X, Y) \neq (X_4, X_3)$ ¹

Now let us build up to formally defining $G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$.

Recall $Z \xrightarrow[\cancel{E}]{T, \text{help}} Y$ from Definition 3.10 for three player games. We shall generalize this to multiplayer games.

Definition 6.1. Definition of player Z not helping player Y in T with respect to E in.

- (i) A κ -player game G satisfies $Z \xrightarrow[\cancel{\kappa, \text{help}}]{T^Y} (Y, E_Y, T^Y)$ (Read: player Z doesn't “help” player Y in the restricted tree, T^Y , of player Y with respect to the set E_Y of positions) iff Y and Z are (different) players of the game G and the following holds:

If $q^r \in T \setminus \text{ORD}_{Y,E}^{\kappa, T}$ and if $\exists z$ such that $q^r(z) \in T \setminus \text{ORD}_{Y,E}^{\kappa, T}$, then in the game G player Z may only play m such that $q^r(m) \in T \setminus \text{ORD}_{Y,E}^{\kappa, T}$.

- (ii) $(Z, E_Z, T^Z) \xrightarrow[\cancel{\kappa, \text{help}}]{\kappa, \text{help}} (Y, E_Y, T^Y)$ abbreviates: $Z \xrightarrow[\cancel{\kappa, \text{help}}]{\kappa, \text{help}} (Y, E_Y, T^Y)$ and $Y \xrightarrow[\cancel{\kappa, \text{help}}]{\kappa, \text{help}} (Z, E_Z, T^Z)$.

Notice $Z \xrightarrow[\cancel{\kappa, \text{help}}]{\kappa, \text{help}} (Y, E_Y, T^Y)$ is our new notation for $Z \xrightarrow[\cancel{E}]{T, \text{help}} Y$ from Definition 3.10. , (Definition 6.1)

Definition 6.2. Definition of the canonical four-player biased open games.

- (i) Let $D_1, D_2,$ and $D_3,$ be pairwise perpendicular sets of positions and let

¹ So “ $X_4 \xrightarrow[\cancel{\text{help}}]{\text{help}} X_3$ ” is not one of the conditions.

$\{X_1, X_2, X_3, X_4\} = \{I, II, III, IV\}$. which players respectively have

$O(D_1)$, $O(D_2)$, and $O(D_3)$ as payoff sets.

Inductively define T^{4, X_i, D_i} as follows:

(a) T^{4, X_0, D_0} = entire game tree.

(b) $T^{4, X_{i+1}, D_{i+1}} = \left\{ \Gamma \in T^{4, X_i, D_i} \mid \forall q \subseteq \Gamma \text{ ORD}_{X_{i+1}, D_{i+1}}^{4, T^{4, X_i, D_i}}(q) \uparrow \right\}$.

When 4 and D_i are clear from the context, we write T^{X_i} for T^{4, X_i, D_i}

We define $G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$ to be the four-player biased open game in

which:

○ player X_i has payoff set $O(D_i)$ for $1 \leq i \leq 3$,

○ G satisfies $(X_1, D_1, T^{X_1}) \xrightarrow{4, \text{help}} (X_2, D_2, T^{X_2})$,

$(X_2, D_2, T^{X_2}) \xrightarrow{4, \text{help}} (X_3, D_3, T^{X_3})$, $(X_1, D_1, T^{X_1}) \xrightarrow{4, \text{help}} (X_3, D_3, T^{X_3})$, and

○ $X_4 \xrightarrow{4, \text{help}} (X_1, D_1, T^{X_1})$, and $X_4 \xrightarrow{4, \text{help}} (X_2, D_2, T^{X_2})$.²

(ii) A four player game G is a *canonical four-player biased open game* if G is

$G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$ for some pairwise perpendicular sets D_1, D_2 , and

D_3 , and for some permutation X_1, X_2, X_3, X_4 of players I, II, III, IV.

, (Definition 6.2)

² Other than a player being allowed to help oneself, only player X_4 , the player with the complementary (possibly non-open) payoff, may try to help another player and moreover player X_4 may only try to help player X_3 , the last player with designated open payoff. Helping and/or not helping X_4 is not defined since he possibly doesn't have open payoff.

Theorem 6.3. All canonical four-player biased open games are determined, i.e. for $\{X_1, X_2, X_3, X_4\} = \{I, II, III, IV\}$ and for any pairwise perpendicular sets $D_1, D_2,$ and $D_3,$ of positions, $G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$ is determined. That is, we have the determinacy of any four-player biased open game G with players $X_1, X_2,$ and X_3 having open payoff sets and in which G satisfies the following:

$$(X_1, D_1, T^{X_1}) \xleftrightarrow{\text{4, help}} (X_2, D_2, T^{X_2}), (X_2, D_2, T^{X_2}) \xleftrightarrow{\text{4, help}} (X_3, D_3, T^{X_3}),$$

$$(X_1, D_1, T^{X_1}) \xleftrightarrow{\text{4, help}} (X_3, D_3, T^{X_3}), X_4 \xrightarrow{\text{help}} (X_1, D_1, T^{X_1}), \text{ and}$$

$$X_4 \xrightarrow{\text{help}} (X_2, D_2, T^{X_2}).$$

Here we drop the D_i 's and the T^{X_i} 's (from the non-helping conditions) since these are clear from the context. We drop the D_i 's from $\text{ORD}_{X_i, D_i}^{4, T^{X_{i-1}}, D_{i-1}}$ since these are also clear from the context.

Proof. Let $D_1, D_2,$ and $D_3,$ be pairwise perpendicular sets of positions. Let

$$\{X_1, X_2, X_3, X_4\} = \{I, II, III, IV\}$$

designate the players and $G = G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$ be as in the hypothesis to the theorem. We will show one of players has a winning strategy.

Let's first summarize which non-helping conditions are used to construct winning strategies for which players. When $\langle \rangle \in \text{ORD}_{X_1}^{4, T^{X_0}}$, no restrictions shall be used to define a winning strategy for player X_1 . When $\langle \rangle \in \text{ORD}_{X_2}^{4, T^{X_1}}$, we shall use the non-helping conditions $X_3 \xrightarrow{\text{help}} X_1$ and $X_4 \xrightarrow{\text{help}} X_1$ to define a winning strategy for player X_2 .

When $\langle \rangle \in \text{ORD}_{X_3}^{4, T^{X_2}}$, we shall use the non-helping conditions $X_1 \not\rightarrow_{\text{help}} X_2$,

$X_4 \not\rightarrow_{\text{help}} X_1$, and $X_4 \not\rightarrow_{\text{help}} X_2$ to define a winning strategy for player X_3 . When

$\langle \rangle \in T^{X_3} = T^{X_2} \setminus \text{ORD}_{X_3}^{4, T^{X_2}}$ (i.e. when the $\langle \rangle$ has no ordinal value for any player), we

shall use that G satisfies that no pair of X_1, X_2 , or X_3 may help one another (i.e.

$X_1 \not\rightarrow_{\text{help}} X_2$, $X_2 \not\rightarrow_{\text{help}} X_3$, and $X_1 \not\rightarrow_{\text{help}} X_3$) to define a winning strategy for player

X_4 .³

When $\langle \rangle \in \text{ORD}_{X_i}^{4, T^{X_{i-1}}}$, we shall want to inductively show $\overset{r}{q} \in \text{ORD}_{X_i}^{4, T^{X_{i-1}}}$ for any legal

play $\overset{r}{q}$ that is according to X_i 's strategy. To do this, we shall first show $\overset{r}{q} \in T^{X_1}$, then

show $\overset{r}{q} \in T^{X_2, L}$, and finally show $\overset{r}{q} \in T^{X_{i-1}}$. In doing this for the cases in which $i \neq 1$, a

particular argument is repeated, which we now present:

Sup/Non-Helping (Sup/NH) Lemma⁴ If $\overset{r}{p} \in T^{X_i}$, $\forall^L m [\overset{r}{p}(m) \in T^{X_{i-1}}]$, U is a player

different than player X_i such that $U \not\rightarrow_{\text{help}} X_i$, and if it is player U 's turn to move at

position $\overset{r}{p}$, then $\forall^L m [\overset{r}{p}(m) \in T^{X_i}]$.

³ When $\langle \rangle$ has restricted ordinal for a player P , we need to be concerned that some player may make a move resulting in a position with (possibly restricted) ordinal for some "earlier" player (earlier in terms of the sequence X_1, X_2, X_3 , not in terms of the order of play). So when $\langle \rangle$ has restricted ordinal for a player P , then the only players who may help an "earlier" player Q , are players P and Q . We don't care if players help player P or any "later" player.

When $\langle \rangle$ has no restricted ordinal for any player, then only the last player may help and he may help any other player, i.e. no two players from X_1, X_2, X_3 may help one another.

⁴ We call this the Sup/NH Lemma, because its proof uses the usual sup-argument to get that is possible to not help player X_i (when $U \neq X_i$) and then uses $U \not\rightarrow_{\text{help}} X_i$ to get the conclusion holds.

Proof. Suppose $\overset{r}{p} \in T^{X_i}$, $\forall^L m \left[\overset{r}{p}(m) \in T^{X_{i-1}} \right]$, $U \neq X_i$, $U \not\rightarrow_{\text{help}} X_i$, and it is not player X_i 's turn to move. Show $\overset{r}{p}(m) \in T^{X_{i-1}}$ and $\overset{r}{p}(m) \notin \text{ORD}_{X_i}^{4,T^{X_{i-1}}}$ for any legal move m .

Since $\overset{r}{p} \in T^{X_{i-1}}$, $\overset{r}{p} \notin \text{ORD}_{X_i}^{4,T^{X_{i-1}}}$, and since $U \neq X_i$, by the sup-argument

$\exists m \in T_{\overset{r}{p}}^{X_{i-1}} \left[\overset{r}{p}(m) \notin \text{ORD}_{X_i}^{4,T^{X_{i-1}}} \right]$. Therefore, it is possible for player U not to help player

X_i . Since $U \not\rightarrow_{\text{help}} X_i$, for any legal move m by player U , $\overset{r}{p}(m) \notin \text{ORD}_{X_i}^{4,T^{X_{i-1}}}$.

Therefore, since $\forall^L m \left[\overset{r}{p}(m) \in T^{X_{i-1}} \right]$ (see the Lemma's hypothesis,

$\forall^L m \left[\overset{r}{p}(m) \in T^{X_i} = T^{X_{i-1}} \setminus \text{ORD}_{X_i}^{4,T^{X_{i-1}}} \right]$. , (Sup/NH Lemma)

Claim 0. The canonical four-player biased open game $G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$ is determined.

This follows from Claims 1, 2, 3, and 4, below.

Claim 1. If $\langle \rangle \in \text{ORD}_{X_1}^{4,T^{X_0}}$, then player I has a winning strategy for the game G .

Assume $\langle \rangle \in \text{ORD}_{X_1}^{4,T^{X_0}}$. The strategy for player X_1 is to play a move that strictly decreases the value of $\text{ORD}_{X_1}^{4,T^{X_0}}$ until we reach a position with $\text{ORD}_{X_1}^{4,T^{X_0}}$ - value of zero.

Due to the definition of $\text{ORD}_{X_1}^{4,T^{X_0}}$, no move by player X_2 , X_3 , or X_4 can take a position in $\text{ORD}_{X_1}^{4,T^{X_0}}$ to a position outside $\text{ORD}_{X_1}^{4,T^{X_0}}$ or to a position with higher $\text{ORD}_{X_1}^{4,T^{X_0}}$ - value.

Therefore any position completing a later inning will have a strictly lower

$\text{ORD}_{X_1}^{4,T^{X_0}}$ - value than positions from earlier innings with nonzero $\text{ORD}_{X_1}^{4,T^{X_0}}$ - value (strictly lower, due to player X_1 's move). Consequently, since there is no infinite strictly decreasing sequence of ordinals, eventually a position is reached with $\text{ORD}_{X_1}^{4,T^{X_0}}$ - value of zero, so that player X_1 wins. , (Claim 1)

Claim 2 If $\langle \rangle \in \text{ORD}_{X_2}^{4,T^{X_1}}$ (so that $\langle \rangle \notin \text{ORD}_{X_1}^{4,T^{X_0}}$) and if G satisfies $X_3 \xrightarrow{\text{help}} X_1$ and $X_4 \xrightarrow{\text{help}} X_1$, then player II has a winning strategy for G .

Assume $\langle \rangle \in \text{ORD}_{X_2}^{4,T^{X_1}}$ and G satisfies $X_3 \xrightarrow{\text{help}} X_1$ and $X_4 \xrightarrow{\text{help}} X_1$. We will describe a strategy for player X_2 such that positions according to player X_2 's strategy are in $\text{ORD}_{X_2}^{4,T^{X_1}}$ (and in particular in T^{X_1}). Player X_2 's strategy will be to play a move that strictly decreases the value of $\text{ORD}_{X_2}^{4,T^{X_1}}$ until we reach a position with $\text{ORD}_{X_2}^{4,T^{X_1}}$ - value of zero. We shall first show that no move by players X_3 or X_4 can take a position in $\text{ORD}_{X_2}^{4,T^{X_1}}$ to a position outside $\text{ORD}_{X_2}^{4,T^{X_1}}$ or to a position with higher $\text{ORD}_{X_2}^{4,T^{X_1}}$ - value. Then we show the same holds for player X_1 . Finally, we provide the details about player X_2 's winning strategy. Since $\langle \rangle \in \text{ORD}_{X_2}^{4,T^{X_1}}$, $\langle \rangle \in T^{X_1}$.

Consider an arbitrary position $\overset{1}{p}$ such that $\overset{1}{p} \in \text{ORD}_{X_2}^{4,T^{X_1}}$ (and therefore $\overset{1}{p} \in T^{X_1}$).

Suppose it is neither player X_1 's nor player X_2 's turn to move. Then it is player U 's turn to move where U is either player X_3 or player X_4 . For any legal move m , we first show:

$$(i) \quad m \in T_{\overset{1}{p}}^{X_1}$$

and then show

$$\text{ORD}_{X_2}^{\Gamma^{X_1}}(\bar{p})(m) \downarrow \text{ and } \text{ORD}_{X_2}^{\Gamma^{X_1}}(\bar{p})(m) \leq \text{ORD}_{X_2}^{\Gamma^{X_1}}(\bar{p}).$$

Since $\bar{p} \in T^{X_1}$, $\forall^L m [\bar{p}(m) \in T^{X_0}]$, $U \neq X_1$, $U \not\stackrel{\text{help}}{\rightarrow} X_1$, and since it is not player X_1 's

turn to move, by the Sup/NH Lemma, $\forall^L m [\bar{p}(m) \in T^{X_1}]$.⁵ Therefore, since

$\bar{p} \in \text{ORD}_{X_2}^{4, \Gamma^{X_1}}$ and $U \neq X_2$, we have for any legal move m by player U ,

$$\text{ORD}_{X_2}^{4, \Gamma^{X_1}}(\bar{p})(m) \downarrow \text{ and } \text{ORD}_{X_2}^{4, \Gamma^{X_1}}(\bar{p})(m) \leq \text{ORD}_{X_2}^{4, \Gamma^{X_1}}(\bar{p}) \text{ by the definition of } \text{ORD}_{X_2}^{4, \Gamma^{X_1}}.$$

This completes showing (i) and (ii) for when it is either player X_3 or X_4 's turn to move.

Now suppose it is player X_1 's turn to move. Recall $\bar{p} \in \text{ORD}_{X_2}^{4, \Gamma^{X_1}}$ (and therefore

$\bar{p} \in T^{X_1}$). By the definition of $\text{ORD}_{X_1}^{4, \Gamma^{X_0}}$, $\bar{p}(m) \in T^{X_1}$ for any move m by player X_1 .

Since $\bar{p}(m) \in T^{X_1}$ and $\bar{p} \in \text{ORD}_{X_2}^{4, \Gamma^{X_1}}$, $\text{ORD}_{X_2}^{4, \Gamma^{X_1}}(\bar{p})(m) \downarrow$ and

$$\text{ORD}_{X_2}^{4, \Gamma^{X_1}}(\bar{p})(m) \leq \text{ORD}_{X_2}^{4, \Gamma^{X_1}}(\bar{p}), \text{ for any legal move } m, \text{ by the definition of } \text{ORD}_{X_2}^{4, \Gamma^{X_1}}.$$
⁶

Now suppose it is player X_2 's turn to move. In this case we describe player X_2 's strategy. Recall $\bar{p} \in \text{ORD}_{X_2}^{4, \Gamma^{X_1}}$. If the $\text{ORD}_{X_2}^{4, \Gamma^{X_1}}(\bar{p}) > 0$, then by the definition of $\text{ORD}_{X_2}^{4, \Gamma^{X_1}}$,

⁵ Since $\bar{p} \in T^{X_1}$ (i.e. $\bar{p} \notin \text{ORD}_{X_1}^{4, \Gamma^{X_0}}$) and since $U \neq X_2$, by the usual sup-argument $\bar{p}(m) \notin \text{ORD}_{X_1}^{4, \Gamma^{X_0}}$ for some move $m \in T_p^{X_1}$. Therefore it is possible for player U not to help player X_1 . Since $U \not\stackrel{\text{help}}{\rightarrow} X_1$, for any legal move m by player U we have $\bar{p}(m) \notin \text{ORD}_{X_1}^{4, \Gamma^{X_0}}$.

⁶ Player X_1 couldn't make a move resulting in a position outside of T^{X_1} by the definition of $\text{ORD}_{X_1}^{4, \Gamma^{X_0}}$ whereas players X_3 and X_4 also couldn't play such a move but due to a different reason: the (non-helping) rules. All these players couldn't make a move resulting in a position outside $\text{ORD}_{X_2}^{4, \Gamma^{X_1}}$ by the definition of $\text{ORD}_{X_2}^{4, \Gamma^{X_1}}$.

there exists a move $m \in T_p^{X_1}$ such that $\text{ORD}_{X_2}^{4,T^{X_1}}(\bar{p}(m)) \downarrow$ and

$\text{ORD}_{X_2}^{4,T^{X_1}}(\bar{p}(m)) < \text{ORD}_{X_2}^{4,T^{X_1}}(\bar{p})$; it is player X_2 's strategy to play such an m .

In summary, any legal move by a player X_1 , X_3 , or X_4 takes a position in $\text{ORD}_{X_2}^{4,T^{X_1}}$ to a position in $\text{ORD}_{X_2}^{4,T^{X_1}}$ with no larger $\text{ORD}_{X_2}^{4,T^{X_1}}$ -value and player X_2 's strategy results in his making moves which strictly decreases the $\text{ORD}_{X_2}^{4,T^{X_1}}$ -value until we reach a position with $\text{ORD}_{X_2}^{4,T^{X_1}}$ -value of zero. Since there is no infinite strictly decreasing sequence of ordinals, eventually a position is reached with $\text{ORD}_{X_2}^{4,T^{X_1}}$ -value of zero, so that player X_2 wins. , (Claim 2)

Claim 3. If $\langle \rangle \in \text{ORD}_{X_3}^{4,T^{X_2}}$ (so that $\langle \rangle \notin \text{ORD}_{X_1}^{4,T^{X_0}}$ and $\langle \rangle \notin \text{ORD}_{X_2}^{4,T^{X_1}}$) and if G satisfies

$X_1 \xrightarrow{\text{help}} X_2$, $X_4 \xrightarrow{\text{help}} X_1$, and $X_4 \xrightarrow{\text{help}} X_2$, then player X_3 has a winning

strategy.

Assume $\langle \rangle \in \text{ORD}_{X_3}^{4,T^{X_2}}$ and G satisfies $X_1 \xrightarrow{\text{help}} X_2$, $X_4 \xrightarrow{\text{help}} X_1$, and

$X_4 \xrightarrow{\text{help}} X_2$. In Claim 2 we showed that a particular strategy for player X_2 is a

winning strategy. Now we show that a particular strategy for player X_3 is a winning

strategy. The proof is almost the same as that of Claim 2, with player X_3 taking on the

role that player X_2 had in Claim 2.

We will describe a strategy for player X_3 such that positions according to player

X_3 's strategy are in $\text{ORD}_{X_3}^{4,T^{X_2}}$ (and in particular in T^{X_2}). Player X_3 's strategy will be to play a move that strictly decreases the value of $\text{ORD}_{X_3}^{4,T^{X_2}}$ until we reach a position with $\text{ORD}_{X_3}^{4,T^{X_2}}$ - value of zero. We shall first show that no move by players X_1 , X_2 , or X_4 can take a position in $\text{ORD}_{X_3}^{4,T^{X_2}}$ to a position outside $\text{ORD}_{X_3}^{4,T^{X_2}}$, or to a position with higher $\text{ORD}_{X_3}^{4,T^{X_2}}$ - value. Then we provide the details of player X_3 's winning strategy. Since $\langle \rangle \in \text{ORD}_{X_3}^{4,T^{X_2}}$, $\langle \rangle \in T^{X_2}$.

Consider an arbitrary position $\overset{\Gamma}{p}$ such that $\overset{\Gamma}{p} \in \text{ORD}_{X_3}^{4,T^{X_2}}$ (and therefore $\overset{\Gamma}{p} \in T^{X_2} \subseteq T^{X_1}$). Suppose it is not player X_3 's turn to move. Then it is player U 's turn to move where $U \neq X_3$. For any legal move m , we first show

$$(i) \quad m \in T_p^{X_1},$$

then show

$$(ii) \quad m \in T_p^{X_2},$$

and finally show

$$(iii) \quad \text{ORD}_{X_3}^{4,T^{X_2}}(\overset{\Gamma}{p})(m) \downarrow \text{ and } \text{ORD}_{X_3}^{4,T^{X_2}}(\overset{\Gamma}{p})(m) \leq \text{ORD}_{X_3}^{4,T^{X_2}}(\overset{\Gamma}{p}).$$

We shall use the Sup/NH Lemma and the definition of $\text{ORD}_{X_1}^{4,T^{X_0}}$ when it is player X_1 's turn to move to show (i). We shall use the Sup/NH Lemma and the definition of $\text{ORD}_{X_2}^{4,T^{X_1}}$ when it is player X_2 's turn to move to show (ii).

Recall it is player U 's (not X_3 's) turn to move and $\overset{\Gamma}{p} \in \text{ORD}_{X_3}^{4,T^{X_2}}$ so that $\overset{\Gamma}{p} \in T^{X_1}$. If $U = X_1$, $m \in T_p^{X_1}$ for any move m by the definition of ORD_{X_1} . Also if $U \neq X_1$, then

$\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_1} \right]$ by the Sup/NH Lemma, since: $\overset{\mathbf{r}}{p} \in T^{X_1}$, $\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_0} \right]$,
 $U \not\stackrel{\text{help}}{\rightarrow} X_1$, and it is not player X_1 's turn to move. Thus, in either case, we have
 shown (i), i.e. $\forall^L m \left[m \in T_{\overset{\mathbf{r}}{p}}^{X_1} \right]$. Since $\overset{\mathbf{r}}{p} \in T^{X_2}$ and $\forall^L m \left[m \in T_{\overset{\mathbf{r}}{p}}^{X_1} \right]$, if $U = X_2$,
 $\overset{\mathbf{r}}{p}(m) \notin \text{ORD}_{X_2}^{4, T^{X_1}}$ for any move m by the definition of $\text{ORD}_{X_2}^{4, T^{X_1}}$; i.e. $\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_2} \right]$.
 Also if $U \neq X_2$, then $\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_2} \right]$ by the Sup/NH Lemma, since: $\overset{\mathbf{r}}{p} \in T^{X_2}$,
 $\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_2} \right]$, $U \not\stackrel{\text{help}}{\rightarrow} X_2$, and it is not player X_2 's turn to move. Thus, in
 either case, we have shown (ii), i.e. $\forall^L m \left[m \in T_{\overset{\mathbf{r}}{p}}^{X_2} \right]$. Recall $\overset{\mathbf{r}}{p} \in \text{ORD}_{X_3}^{4, T^{X_2}}$ and it is player
 U 's (not X_3 's) turn to move so for any legal move m , $\text{ORD}_{X_3}^{4, T^{X_2}}(\overset{\mathbf{r}}{p}(m)) \downarrow$ and
 $\text{ORD}_{X_3}^{4, T^{X_2}}(\overset{\mathbf{r}}{p}(m)) \leq \text{ORD}_{X_3}^{T^{X_2}}(\overset{\mathbf{r}}{p})$ by the definition of $\text{ORD}_{X_3}^{4, T^{X_2}}$. This completes showing
 (i), (ii), and (iii) when it is either player X_1 , X_2 or X_4 's turn to move.

Now suppose it is player X_3 's turn to move. In this case, we describe player X_3 's
 strategy. Recall $\overset{\mathbf{r}}{p} \in \text{ORD}_{X_3}^{4, T^{X_2}}$. If $\text{ORD}_{X_3}^{4, T^{X_2}}(\overset{\mathbf{r}}{p}) > 0$, then by the definition of $\text{ORD}_{X_3}^{4, T^{X_2}}$
 there exists a move $m \in T_{\overset{\mathbf{r}}{p}}^{X_2}$ such that $\text{ORD}_{X_3}^{4, T^{X_2}}(\overset{\mathbf{r}}{p}(m)) \downarrow$ and
 $\text{ORD}_{X_3}^{4, T^{X_2}}(\overset{\mathbf{r}}{p}(m)) < \text{ORD}_{X_3}^{4, T^{X_2}}(\overset{\mathbf{r}}{p})$; it is player X_3 's strategy to play such an m .

In summary, any legal move by a player X_1 , X_2 , or X_4 takes a position in $\text{ORD}_{X_3}^{4, T^{X_2}}$ to
 a position in $\text{ORD}_{X_3}^{4, T^{X_2}}$ with no larger $\text{ORD}_{X_3}^{4, T^{X_2}}$ -value and player X_3 's strategy results in
 his making moves which strictly decreases the $\text{ORD}_{X_3}^{4, T^{X_2}}$ -value until we reach a position
 with $\text{ORD}_{X_3}^{4, T^{X_2}}$ -value of zero. Since there is no infinite strictly decreasing sequence of

ordinals, eventually a position is reached with $\text{ORD}_{X_3}^{4,T^{X_2}}$ - value of zero, so that player X_3 wins the game. , (Claim 3)

Claim 4. If $\langle \rangle \in T^{X_3} = T^{X_2} \setminus \text{ORD}_{X_3}^{4,T^{X_2}}$ (so that $\langle \rangle \notin \text{ORD}_{X_1}^{4,T^{X_0}}$, $\langle \rangle \notin \text{ORD}_{X_2}^{4,T^{X_1}}$, and $\langle \rangle \notin \text{ORD}_{X_3}^{4,T^{X_2}}$) and if G satisfies that no pair of X_1 , X_2 , and X_3 may help one another (i.e. $X_1 \not\stackrel{\text{help}}{\rightarrow} X_2$, $X_2 \not\stackrel{\text{help}}{\rightarrow} X_3$, and $X_1 \not\stackrel{\text{help}}{\rightarrow} X_3$), then player X_4 has a winning strategy for G .

Assume $\langle \rangle \in T^{X_3} = T^{X_2} \setminus \text{ORD}_{X_3}^{4,T^{X_2}}$ and that no pair of X_1 , X_2 , and X_3 may help one another. We will describe a strategy for player X_4 such that positions according to player X_4 's strategy are in $T^{X_3} = T^{X_2} \setminus \text{ORD}_{X_3}^{4,T^{X_2}}$; then we shall show that the strategy must be a winning strategy for player X_4 . We shall first show that no move by any player U , where $U \neq X_4$, can take a position in $T^{X_3} = T^{X_2} \setminus \text{ORD}_{X_3}^{4,T^{X_2}}$ to a position outside $T^{X_3} = T^{X_2} \setminus \text{ORD}_{X_3}^{4,T^{X_2}}$. Then we describe the strategy for player X_4 . Recall $\langle \rangle \in T^{X_3} = T^{X_2} \setminus \text{ORD}_{X_3}^{4,T^{X_2}}$.

Consider an arbitrary position $\overset{\Gamma}{p}$ such that $\overset{\Gamma}{p} \in T^{X_3}$ (so that $\overset{\Gamma}{p} \in T^{X_3} \subseteq T^{X_2} \subseteq T^{X_1} \subseteq T^{X_0}$). Suppose it is player U 's turn to move, where $U \neq X_4$. For any legal move m we show that

$$(i) \quad m \in T_{\overset{\Gamma}{p}}^{X_1},$$

then show

$$(ii) \quad m \in T_p^{X_2},$$

and finally show

$$(iii) \quad m \in T_p^{X_3}.$$

We shall use the Sup/NH Lemma and the definition of $ORD_{X_1}^{4,T^{X_0}}$ when it is player X_1 's turn to move to show (i). We shall use the Sup/NH Lemma and the definition of $ORD_{X_2}^{4,T^{X_1}}$ when it is player X_2 's turn to move to show (ii). We shall use the Sup/NH Lemma and the definition of $ORD_{X_3}^{4,T^{X_2}}$ to show (iii).

Recall it is player U 's (not X_4 's) turn to move and $\frac{r}{p} \in T^{X_3}$ so that $\frac{r}{p} \in T^{X_1}$. If $U = X_1$, $m \in T_p^{X_1}$ for any move m by the definition of ORD_{X_1} . Also if $U \neq X_1$, then $\forall^L m \left[\frac{r}{p}(m) \in T^{X_1} \right]$ by the Sup/NH Lemma, since: $\frac{r}{p} \in T^{X_1}$, $\forall^L m \left[\frac{r}{p}(m) \in T^{X_0} \right]$, $U \xrightarrow{\text{help}} X_1$, and it is not player X_1 's turn to move. Thus, in either case, we have shown (i), i.e. $\forall^L m \left[m \in T_p^{X_1} \right]$. Just as we have now shown that $\forall^L m \left[m \in T_p^{X_1} \right]$, the same argument gives that $\forall^L m \left[m \in T_p^{X_2} \right]$ and $\forall^L m \left[m \in T_p^{X_3} \right]$. To show $\forall^L m \left[m \in T_p^{X_2} \right]$ recall it is player U 's turn to move, $\frac{r}{p} \in T^{X_2}$, and $\forall^L m \left[m \in T_p^{X_1} \right]$. Therefore, if $U = X_2$, $m \in T_p^{X_2}$ for any (legal) move m by the definition of $ORD_{X_2}^{4,T^{X_1}}$. Also if $U \neq X_2$, then $\forall^L m \left[\frac{r}{p}(m) \in T^{X_2} \right]$ by the Sup/NH Lemma, since: $\frac{r}{p} \in T^{X_2}$, $\forall^L m \left[\frac{r}{p}(m) \in T^{X_1} \right]$, $U \xrightarrow{\text{help}} X_2$, and it is not player X_2 's turn to move. Thus, in either case, we have

shown (ii), i.e. $\forall^L m \left[m \in T_p^{X_2} \right]$. Finally the same argument gives that $\forall^L m \left[m \in T_p^{X_3} \right]$.⁷

This completes showing (i), (ii), and (iii) when it is either player X_1 , X_2 , or X_3 's turn to move.

Suppose it is player X_4 's turn to move. Since $\frac{\Gamma}{p} \in T^{X_2}$ and $\frac{\Gamma}{p} \notin \text{ORD}_{X_3}^{4, T^{X_2}}$, by the usual sup-argument, $\exists m \in T_p^{X_2}$ $\frac{\Gamma}{p}(m) \notin \text{ORD}_{X_3}^{4, T^{X_2}}$. The strategy for player X_4 is to play such an m , (i.e. that $\frac{\Gamma}{p}(m) \in T^{X_3} = T^{X_2} \setminus \text{ORD}_{X_3}^{4, T^{X_2}}$).⁸

We now explain why player X_4 's strategy is a winning strategy. We have shown that any legal move by a player X_1 , X_2 , or X_3 takes a position in T^{X_3} to a position in T^{X_3} and any move according to player X_4 's strategy takes a position in T^{X_3} to a position in T^{X_3} . Thus for any play \bar{y} according to player X_4 's strategy, $\bar{y}(n) \in T^{X_3}$ for every n . Therefore, since $T^{X_3} \subseteq T^{X_2} \subseteq T^{X_1} \subseteq T^{X_0}$ we have $\bar{y}(n) \notin \text{ORD}_{X_1}^{4, T^{X_0}}$, $\bar{y}(n) \notin \text{ORD}_{X_2}^{4, T^{X_1}}$, and $\bar{y}(n) \notin \text{ORD}_{X_3}^{4, T^{X_2}}$. Hence \bar{y} is a loss for all players X_1 , X_2 , or X_3 and is therefore a win for player X_4 . , (Claim 4)

Thus Claim) follows from Claims 1, 2, 3, and 4. Thus the canonical four-player

⁷ Recall it is player U 's turn to move, $\frac{\Gamma}{p} \in T^{X_3}$, and $\forall^L m \left[m \in T_p^{X_2} \right]$. If $U = X_3$, $m \in T_p^{X_2}$ for any move m by the definition of $\text{ORD}_{X_3}^{4, T^{X_2}}$. Also if $U \neq X_3$, $\forall^L m \left[\frac{\Gamma}{p}(m) \in T_p^{X_3} \right]$ by the Sup/NH Lemma since: $\frac{\Gamma}{p} \in T^{X_3}$, $\forall^L m \left[m \in T_p^{X_3} \right]$, $U \neq X_3$, $U \xrightarrow{\text{help}} X_3$, and it is not player X_3 's turn to move. Thus, in either case we have shown (iii), i.e. $\forall^L m \left[m \in T_p^{X_3} \right]$.

⁸ Since $\frac{\Gamma}{p}(m) \in T^{X_2} \subseteq T^{X_1}$, we have $\frac{\Gamma}{p}(m) \notin \text{ORD}_{X_1}^{4, T^{X_0}}$, $\frac{\Gamma}{p}(m) \notin \text{ORD}_{X_2}^{4, T^{X_1}}$ and $\frac{\Gamma}{p}(m) \notin \text{ORD}_{X_3}^{4, T^{X_2}}$.

biased open game $G_{(X_1, X_2, X_3, X_4)}(D_1, D_2, D_3)$ is determined. , (Theorem 6.3)

Now let us look at a generalized proof of a canonical multiplayer biased open game in which there are $(n + 1)$ players.

Definition 6.4: Definition of canonical $(n + 1)$ -player biased open games.

(i). Let $n \geq 2$. Let $\overset{1}{A} = (A_1, A_2, A_3, L, A_{n-1}, A_n)$ be a sequence of pairwise perpendicular sets of positions and let $\overset{1}{X} = (X_1, X_2, X_3, L, X_n, X_{n+1})$ be the players one through $(n + 1)$ in any order.

Let $N = n + 1$. Inductively define T^{n+1, X_i, D_i} as follows:

T^{N, X_0, D_0} is the entire game tree.

$$T^{N, X_{i+1}, D_{i+1}} = \left\{ \overset{r}{p} \in T^{N, X_i, D_i} \mid \forall \overset{r}{q} \subseteq \overset{r}{p} \text{ ORD}_{X_{i+1}, D_{i+1}}^{N, T^{N, X_i, D_i}} \right\}.$$

Typically D_i and $N = n + 1$ are clear from the context, in which case we write T^{X_i} for

T^{n+1, X_i, D_i} . Also we write $\text{ORD}_{X_{i+1}}^{T^{X_i}}$ for $\text{ORD}_{X_{i+1}, D_{i+1}}^{N, T^{N, X_i, D_i}}$

Let Define $G_{\overset{r}{X}}(\overset{r}{A})$ to be the $(n+1)$ -player biased open game in which player X_i has

payoff set $O(D_i)$ for $1 \leq i \leq n$, and in which G satisfies:

- $X_i \xrightarrow{\text{help}} (X_j, D_j, T^j)$ for all $i \leq n$ and $j \leq n$ such that $i \neq j$ and
- $X_{n+1} \xrightarrow{\text{help}} (X_i, D_i, T^i)$ for all $i < n$.

(Other than a player being allowed to help oneself, only player X_{n+1} , the player with possibly non-open payoff, may try to help X_n , the last player with designated open

payoff. Helping and/or not helping player X_{n+1} is not defined since he possibly doesn't have open payoff.)

(ii) An $(n+1)$ -player game G is a *canonical $(n+1)$ -player biased open game* if G is $G_{\bar{X}}(\bar{D})$ for some sequence $\bar{D} = (D_1, D_2, K, D_{n-1}, D_n)$ of pairwise perpendicular sets of positions and for some sequence $\bar{X} = (X_1, X_2, K, X_n, X_{n+1})$ of players one through $(n+1)$. (Definition 6.4)

Theorem 6.5. Let $n \geq 2$. All $i < n$, that is all canonical $(n+1)$ -player biased open games are determined.

Proof. Let $n \geq 2$, let $\bar{D} = (D_1, D_2, K, D_{n-1}, D_n)$ be a sequence of pairwise perpendicular sets of positions, and let $\bar{X} = (X_1, X_2, K, X_n, X_{n+1})$ be the players one through $(n+1)$ in any order as in Definition 6.3. Let G be a $(n+1)$ -player game in which player X_i has payoff set $O(D_i)$ if $1 \leq i \leq n$. We will eventually show that G is determined if G satisfies: $X_i \not\rightarrow^{\text{help}} (X_j, D_j, T^j)$ for all $i \leq n$ and $j \leq n$ such that $i \neq j$ and, $X_{n+1} \not\rightarrow^{\text{help}} (X_i, D_i, T^i)$ for all $i < n$. Here we drop the D_i 's and T^i 's since these are clear from the context.

As usual, we shall use ordinals of position, $\text{ORD}_{X_i}^{T^{X_{i-1}}}$, for various players together with the non-helping conditions to construct a winning strategy. In general we should not expect the non-helping conditions to exclude a given player from being helped by himself

or by the player who will have a winning strategy.⁹ Therefore, if $\langle \rangle \in \text{ORD}_{X_i}^{T^{X_{i-1}}}$ let us say that *essentially nobody* helps player U if the only players allowed to help player U are players U and X_i (i.e. for every player X, $X \not\rightarrow_{\text{help}} U$ or $X \in \{U, X_i\}$) Let's first summarize which non-helping conditions are used to construct winning strategies for which players. When $\langle \rangle \in \text{ORD}_{X_i}$, we shall define a winning strategy for player X_i without using any restrictions. When $2 \leq i \leq n$ and $\langle \rangle \in \text{ORD}_{X_i}^{T^{X_{i-1}}}$, we shall use the non-

helping conditions that essentially nobody can help players X_1, X_2, \dots, X_{i-1} , i.e.

$X_j \not\rightarrow_{\text{help}} X_k$ for any $k < i$ and any $j \notin \{k, i\}$ to define a winning strategy for player

X_i .¹⁰ When $\langle \rangle \in T^{X_n} = T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{T^{X_{n-1}}}$ (i.e. when the $\langle \rangle$ has no ordinal for any players),

we shall use the non-helping conditions (i.e. $X_i \not\rightarrow_{\text{help}} X_j$ for all $i \leq n$ and all $j \leq n$

such that $j \neq i$)¹¹ which state no pair of players X_1, X_2, \dots, X_n may help each other to

define a winning strategy for player X_{n+1} .

As in Theorem 6.3, we shall also use the following:

Sup/Non-Helping (Sup/NH) Lemma: If $\frac{r}{p} \in T^{X_i}$, $\forall^L m \left[\frac{r}{p}(m) \in T^{X_{i-1}} \right]$, U is a player

different than player X_i such that $U \not\rightarrow_{\text{help}} X_i$, and if it is player U's turn to move at

position $\frac{r}{p}$, then $\forall^L m \left[\frac{r}{p}(m) \in T^{X_i} \right]$.

⁹ The winning strategy for such a player will include that he doesn't help the other player and therefore a non-helping rule for this is not needed.

¹⁰ Analogous to footnote #1, when $\langle \rangle$ has restricted ordinal for a player X_i ($2 \leq i \leq n$), then the only players who may help an "earlier" player X_k ($k < i$) are players X_i and X_k .

¹¹ When $\langle \rangle$ has no restricted ordinal for any player, then only the last player may help and he may help any other player.

The proof of the Sup/NH Lemma is the same as that presented in Theorem 6.1. Also a description of the manner in which the Lemma is used is presented in the paragraph preceding its statement in Theorem 6.3.

Claim 0. G is determined if G satisfies $X_i \xrightarrow{\text{help}} (X_j, D_j, T^j)$ for all $i \leq n$ and $j \leq n$ such that $i \neq j$, and $X_{n+1} \xrightarrow{\text{help}} (X_i, D_i, T^i)$ for all $i < n$

This follows from the Claims 1 through $(n + 1)$ below

Claim 1. If $\langle \rangle \in \text{ORD}_{X_1}$, then player X_1 has a winning strategy for the game G .

Assume $\langle \rangle \in \text{ORD}_{X_1}$. The strategy for player X_1 is to play a move that strictly decreases the value of ORD_{X_1} until we reach a position with ORD_{X_1} -value of zero. Due to the definition of ORD_{X_1} , no move by a player X_2, X_3, L, X_n , or X_{n+1} can take a position in ORD_{X_1} to a position outside ORD_{X_1} or with higher ORD_{X_1} -value. Therefore any position completing a later inning will have a strictly lower ORD_{X_1} -value than positions from earlier innings with nonzero ORD_{X_1} -value. Consequently, since there is no infinite strictly decreasing sequence of ordinals, eventually a position is reached with ORD_{X_1} -value of zero, so that player X_1 wins. , (Claim 1)

Claim 2. If $\langle \rangle \in \text{ORD}_{X_2}^{T^{X_1}}$ (so that $\langle \rangle \notin \text{ORD}_{X_1}$), and G satisfies $X_i \xrightarrow{\text{help}} X_1$, for $i \in \{3, 4, 5, \dots, n+1\}$, then player II has a winning strategy for G .

Assume $\langle \rangle \in \text{ORD}_{X_2}^{T^{X_1}}$ and G satisfies $X_i \xrightarrow{\text{help}} X_1$ for $i \in \{3, 4, 5, K, n+1\}$. We will describe a strategy for player X_2 such that positions according to player X_2 's strategy are in $\text{ORD}_{X_2}^{T^{X_1}}$ (and therefore in T^{X_1}). Player X_2 's strategy will be to play a move that strictly decreases the value of $\text{ORD}_{X_2}^{T^{X_1}}$ until we reach a position with $\text{ORD}_{X_2}^{T^{X_1}}$ - value of zero. We shall first show that no move by players X_3, X_4, L , or X_{n+1} can take a position in $\text{ORD}_{X_2}^{T^{X_1}}$ to a position outside $\text{ORD}_{X_2}^{T^{X_1}}$ or with higher $\text{ORD}_{X_2}^{T^{X_1}}$ - value. Then we will show the same for player X_1 . Finally we shall provide more details about player X_2 's winning strategy. Since $\langle \rangle \in \text{ORD}_{X_2}^{T^{X_1}}, \langle \rangle \in T^{X_1}$.

Consider an arbitrary position $\overset{\mathbf{r}}{p}$ such that $\overset{\mathbf{r}}{p} \in \text{ORD}_{X_2}^{T^{X_1}}$ (and therefore $\overset{\mathbf{r}}{p} \in T^{X_1}$).

Suppose it is player X_j 's turn to move where $3 \leq j \leq n+1$ (so that it is neither player X_1 nor player X_2 's turn to move, $X_j \neq X_1$, and $X_j \neq X_2$). For any legal move m , we first show

$$(i) \quad m \in T_{\overset{\mathbf{r}}{p}}^{X_j}$$

and then show

$$(ii) \quad \text{ORD}_{X_2}^{T^{X_1}}(\overset{\mathbf{r}}{p}(m)) \downarrow \text{ and } \text{ORD}_{X_2}^{T^{X_1}}(\overset{\mathbf{r}}{p}(m)) \leq \text{ORD}_{X_2}^{T^{X_1}}(\overset{\mathbf{r}}{p}).$$

Since $\overset{\mathbf{r}}{p} \in T^{X_1}, \forall^L m[\overset{\mathbf{r}}{p}(m) \in T^{X_0}], X_j \neq X_1, X_j \xrightarrow{\text{help}} X_1$, and since it is not player

X_1 's turn to move, by the Sup/NH Lemma, $\forall^L m[\overset{\mathbf{r}}{p}(m) \in T^{X_1}]$. Therefore, since

$\overset{\mathbf{r}}{p} \in \text{ORD}_{X_2}^{T^{X_1}}$ and $X_j \neq X_2, (j \neq 2)$, we have for any legal move m by player X_j ,

$\text{ORD}_{X_2}^{T^{X_1}}(\overset{\mathbf{r}}{p}(m)) \downarrow \text{ and } \text{ORD}_{X_2}^{T^{X_1}}(\overset{\mathbf{r}}{p}(m)) \leq \text{ORD}_{X_2}^{T^{X_1}}(\overset{\mathbf{r}}{p})$, by the definition of $\text{ORD}_{X_2}^{T^{X_1}}$. This

completes showing (i) and (ii) when it is neither player X_1 nor player X_2 's turn to move.

Now suppose it is player X_1 's turn to move. Recall $\overset{\Gamma}{p} \in \text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$ (and therefore $\overset{\Gamma}{p} \in T^{X_1}$). By the definition of ORD_{X_1} , $\overset{\Gamma}{p}(m) \in T^{X_1}$ for any move m by player X_1 . Since $\overset{\Gamma}{p}(m) \in T^{X_1}$ and $\overset{\Gamma}{p} \in \text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$, $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}(\overset{\Gamma}{p}(m)) \downarrow$ and $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}(\overset{\Gamma}{p}(m)) \leq \text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}(\overset{\Gamma}{p})$, for any legal move m by player X_1 , by the definition of $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$.

Now suppose it is player X_2 's turn to move. In this case we describe player X_2 's strategy. Recall $\overset{\Gamma}{p} \in \text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$. If $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}(\overset{\Gamma}{p}) > 0$, then there exists a move m such that $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}(\overset{\Gamma}{p}(m)) \downarrow$ and $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}(\overset{\Gamma}{p}(m)) < \text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}(\overset{\Gamma}{p})$; it is player X_2 's strategy to play such an m .

In summary, any legal move by any player X_j , where $j \neq 2$, takes a position in $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$ to a position in $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$ with no larger $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$ -value and player X_2 's strategy results in his making moves which strictly decreases the $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$ -value until we reach a position with $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$ -value of zero. Since there is no infinite strictly decreasing sequence of ordinals, eventually a position is reached with $\text{ORD}_{X_2}^{\overset{\Gamma}{X_1}}$ -value of zero, so that player X_2 wins. , (Claim 2)

Claim i. If $\langle \rangle \in \text{ORD}_{X_i}^{\overset{\Gamma}{X_{i-1}}}$ for some i such that $3 \leq i < n$, then player X_i has a winning strategy for G when G satisfies that essentially nobody helps players X_1, X_2, \dots, X_{i-1} , i.e G satisfies $X_j \not\overset{\text{help}}{\rightarrow} X_k$ for any $k < i$ and any $j \notin \{k, i\}$.

(so that $\langle \rangle \notin \text{ORD}_{X_1}, \langle \rangle \notin \text{ORD}_{X_2}^{T^{X_1}}, \mathbf{L}, \langle \rangle \notin \text{ORD}_{X_{i-1}}^{T^{X_{i-2}}}$).

Since $\langle \rangle \notin \text{ORD}_{X_1}, \langle \rangle \notin \text{ORD}_{X_2}^{T^{X_1}}, \mathbf{L}, \langle \rangle \notin \text{ORD}_{X_{i-1}}^{T^{X_{i-2}}}$, assume $\langle \rangle \in \text{ORD}_{X_i}^{T^{X_{i-1}}}$, and G satisfies that essentially nobody helps players X_1, X_2, \dots, X_{i-1} . In Claim 2 we showed that a particular strategy for player X_2 is a winning strategy. Here we will show that a particular strategy for player X_i is a winning strategy. The proof (for Case i) is almost the same as that of Case 2, with player X_i taking on the role that player X_2 had in Case 2. We shall first show that

(*) no move by any player U , where $U \neq X_i$ can take a position in $\text{ORD}_{X_i}^{T^{X_{i-1}}}$ to

a position outside $\text{ORD}_{X_i}^{T^{X_{i-1}}}$ or to a position with higher $\text{ORD}_{X_i}^{T^{X_{i-1}}}$ -value.¹²

Then we describe the (usual) strategy for player X_i and show that it is a winning strategy. Player X_i 's strategy will be to play a move that strictly decreases the value of $\text{ORD}_{X_i}^{T^{X_{i-1}}}$ until we reach a position with $\text{ORD}_{X_i}^{T^{X_{i-1}}}$ -value of zero. Since $\langle \rangle \in \text{ORD}_{X_i}^{T^{X_{i-1}}}$, $\langle \rangle \in T^{X_{i-1}}$.

Consider an arbitrary position $\overset{\mathbf{r}}{p}$ such that $\overset{\mathbf{r}}{p} \in \text{ORD}_{X_i}^{T^{X_{i-1}}}$ (so that

$\overset{\mathbf{r}}{p} \in T^{X_{i-1}} \subseteq T^{X_{i-2}} \subseteq \mathbf{L} T^{X_1} \subseteq T^{X_0}$ and $\overset{\mathbf{r}}{p} \notin \text{ORD}_{X_{i+1}}^{T^{X_{i-2}}}$). Suppose it is player U 's turn to move

where $U \neq X_i$. For any legal move m , we first show

$$(1) m \in T_{\overset{\mathbf{r}}{p}}^{X_1}, m \in T_{\overset{\mathbf{r}}{p}}^{X_2}, \mathbf{L}, \text{ and } m \in T_{\overset{\mathbf{r}}{p}}^{X_k}, \text{ i.e. } \forall k \leq i-1 \forall^{\mathbf{L}} m [m \in T_{\overset{\mathbf{r}}{p}}^{X_k}],$$

¹² Our first step in showing this is to show that no move by player U , when $U \neq X_i$, can take a position in T^{X_i} to a position outside T^{X_i} . We shall use the Sup-Non-Helping Lemma and the definition of $\text{ORD}_{X_i}^{T^{X_{i-1}}}$ when $k < i$ to show this (first step).

and then show

$$(2) \text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})(m) \downarrow \text{ and } \text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})(m) \leq \text{ORD}_{X_i}^{\mathbf{r}}(\bar{p}).$$

First we shall prove (1) by induction on k . (1) trivially holds for $k = 0$

$(\forall^L m [m \in T_{\bar{p}}^{\mathbf{r}^{X_0}}])$. Fix $1 \leq j \leq i-1$, assume as the Induction Hypothesis $\forall^L m [m \in T_{\bar{p}}^{\mathbf{r}^{X_{j-1}}}]$,

and show (1) holds at $k = j$. If $U = X_j$ (recall it is player U 's turn to move), we shall

use both the definition of $\text{ORD}_{X_j}^{\mathbf{r}^{X_{j-1}}}$ ¹³ and the Induction Hypothesis $\forall^L m [m \in T_{\bar{p}}^{\mathbf{r}^{X_{j-1}}}]$ to

get $\forall^L m [\bar{p}(m) \in T^{X_j}]$. If $U \neq X_j$, we shall use the Sup/NH Lemma and the Induction

Hypothesis to get $\forall^L m [\bar{p}(m) \in T^{X_j}]$.

Recall $\bar{p} \in \text{ORD}_{X_i}^{\mathbf{r}^{X_{i-1}}}$ (and therefore $p \in T^{X_{j-1}}$) and it is player U 's turn to move. If

$U = X_j$, then since $\bar{p} \in T^{X_j}$, $\bar{p}(m) \notin \text{ORD}_{X_j}^{\mathbf{r}^{X_{j-1}}}$ for any move m by player X_j by the

definition of $\text{ORD}_{X_j}^{\mathbf{r}^{X_{j-1}}}$ and $\forall^L m [m \in T_{\bar{p}}^{\mathbf{r}^{X_{j-1}}}]$ (by the Induction Hypothesis). Therefore,

$\forall^L m [m \in T_{\bar{p}}^{\mathbf{r}^{X_j}}]$. Also if $U \neq X_j$, then by the Sup/NH Lemma $\forall^L m [\bar{p}(m) \in T^{X_j}]$ since:

$\bar{p} \in T^{X_j}$, $\forall^L m [m \in T_{\bar{p}}^{\mathbf{r}^{X_{j-1}}}]$ (by the Induction Hypothesis), $U \xrightarrow{\text{help}} X_j$, and it is not

player X_j 's turn to make a move. Therefore, in either case, for any legal move m by

player U , $m \in T_{\bar{p}}^{\mathbf{r}^{X_j}}$. Consequently, using induction we have shown

$$(1) \forall k \leq i-1 \forall^L m [m \in T_{\bar{p}}^{\mathbf{r}^{X_k}}];$$

in particular we have $m \in T_{\bar{p}}^{\mathbf{r}^{X_{i-1}}}$.

¹³ $\text{ORD}_{X_j}^{\mathbf{r}^{X_{j-1}}}$ makes sense since $1 \leq j \leq i-1$.

Now we show that (2) $\text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})(m) \downarrow$ and $\text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})(m) \leq \text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})$ holds.

Since $\bar{p} \in \text{ORD}_{X_i}^{\mathbf{r}}$ and $\forall^L m [m \in T_{\bar{p}}^{\mathbf{r}}]$ (by (1)), we have for any legal move m ,

$\text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})(m) \downarrow$ and $\text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})(m) \leq \text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})$, by the definition of $\text{ORD}_{X_i}^{\mathbf{r}}$.

This completes our proof of (*).

Now suppose it is player X_i 's turn to move. In this case, we describe player X_i 's strategy. Recall $\bar{p} \in \text{ORD}_{X_i}^{\mathbf{r}}$. If $\text{ORD}_{X_i}^{\mathbf{r}}(\bar{p}) > 0$ then there exists a move $m \in T_{\bar{p}}^{\mathbf{r}}$ such that $\text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})(m) \downarrow$ and $\text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})(m) < \text{ORD}_{X_i}^{\mathbf{r}}(\bar{p})$; it is player X_i 's strategy to play such an m .

In summary, any legal move by any player X_j , where $j \neq i$, takes a position in $\text{ORD}_{X_i}^{\mathbf{r}}$ to a position in $\text{ORD}_{X_i}^{\mathbf{r}}$ with no larger $\text{ORD}_{X_i}^{\mathbf{r}}$ -value and player X_i 's strategy results in his making moves which strictly decreases the $\text{ORD}_{X_i}^{\mathbf{r}}$ -value until we reach a position with $\text{ORD}_{X_i}^{\mathbf{r}}$ -value of zero. Since there is no infinite strictly decreasing sequence of ordinals, eventually a position is reached with $\text{ORD}_{X_i}^{\mathbf{r}}$ -value of zero, so that player X_i wins. , (Claim i)

Case (n + 1). If $\langle \rangle \in T^{X_n} = T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{\mathbf{r}}$ (so that $\langle \rangle \in T^{X_i}$ and $\langle \rangle \notin \text{ORD}_{X_i}^{\mathbf{r}}$ for all

$i \leq n$) and if G satisfies $X_i \xrightarrow{\text{help}} X_j$ for all $i \leq n$ and for all $j \leq n$ such that $i \neq j$, then

player (n + 1) has a winning strategy for the game G .

Assume $\langle \rangle \in T^{X_n} = T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{\mathbf{r}}$ and if G satisfies $X_i \xrightarrow{\text{help}} X_j$ for all $i \leq n$ and

for all $j \leq n$ such that $i \neq j$. We will describe a strategy for player X_{n+1} such that positions according to player X_{n+1} 's strategy are in $T^{X_n} = T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{T^{X_{n-1}}}$; then we shall show that the strategy must be a winning strategy for player X_{n+1} . We shall first show that

(**) no move by any player U , where $U \neq X_{n+1}$, can take a position in $T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{T^{X_{n-1}}}$ to a position outside $T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{T^{X_{n-1}}}$.

Then we describe the strategy for player X_{n+1} . Recall $\langle \rangle \in T^{X_n} = T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{T^{X_{n-1}}}$.

Consider an arbitrary position $\overset{\mathbf{r}}{p}$ such that $\overset{\mathbf{r}}{p} \in T^{X_n} = T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{T^{X_{n-1}}}$ (so that $p \in T^{X_n} \subseteq T^{X_{n-1}}L \subseteq T^{X_1} \subseteq T^{X_0}$). Suppose it is player U 's turn to move, where $U \neq X_{n+1}$.

For any legal move m we prove

$$(3) \ m \in T_{\overset{\mathbf{r}}{p}}^{X_1}, \ m \in T_{\overset{\mathbf{r}}{p}}^{X_2}, \ L, \ m \in T_{\overset{\mathbf{r}}{p}}^{X_{n-1}}, \ \text{and} \ m \in T_{\overset{\mathbf{r}}{p}}^{X_n}, \ \text{i.e.} \ \forall k \leq n \ \forall^L m \left[m \in T_{\overset{\mathbf{r}}{p}}^{X_k} \right],$$

by induction on k . The proof of (3) is the same as that of (1) in Case i (k varies through $k < i$ in Case i and k varies through $k < n+1$ here). (3) trivially holds for $k = 0$. Fix

$1 \leq j \leq n$, assume as the Induction Hypothesis $\forall^L m \left[m \in T_{\overset{\mathbf{r}}{p}}^{X_{j-1}} \right]$, and show (3) holds at

$k = j$. If $U = X_j$, we shall use both the definition of $\text{ORD}_{X_j}^{T^{X_{j-1}}}$ and the Induction

Hypothesis $\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_{j-1}} \right]$ to get $\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_j} \right]$. If $U \neq X_j$, we shall use the

Sup/NH Lemma and the Induction Hypothesis to get $\forall^L m \left[\overset{\mathbf{r}}{p}(m) \in T^{X_j} \right]$.

Recall $\overset{\mathbf{r}}{p} \in T^{X_{n-1}}$ and it is player U 's turn to move. Since $\overset{\mathbf{r}}{p} \in T^{X_{n-1}} \subseteq T^{X_{j-1}}$, $\overset{\mathbf{r}}{p} \in T^{X_{j-1}}$.

If $U = X_j$, then since $\overset{\mathbf{r}}{p} \in T^{X_j}$, $\overset{\mathbf{r}}{p}(m) \notin \text{ORD}_{X_j}^{T^{X_{j-1}}}$ for any move m by player X_j by the

definition of $\text{ORD}_{X_j}^{T^{X_{j-1}}}$ and $\forall^L m \left[\overset{\Gamma}{p}(m) \in T^{X_{j-1}} \right]$ (by the Induction Hypothesis).

$T^{X_{j-1}} \setminus \text{ORD}_{X_j}^{T^{X_{j-1}}}$ for any legal move m , i.e. $\forall^L m \left[m \in T^{X_j} \right]$. Also if $U \neq X_j$, then by the Sup/NH Lemma $\forall^L m \left[\overset{\Gamma}{p}(m) \in T^{X_j} \right]$ since: $\overset{\Gamma}{p} \in T^{X_j}$, $\forall^L m \left[m \in T_{\overset{\Gamma}{p}}^{X_{j-1}} \right]$, (by the Induction Hypothesis), $U \xrightarrow{\text{help}} X_j$, and it is not player X_j 's turn to make a move. Therefore, in either case, for any legal move m by player U , $m \in T_{\overset{\Gamma}{p}}^{X_j}$. Consequently, using induction we have shown $\forall k \leq n \forall^L m \left[m \in T_{\overset{\Gamma}{p}}^{X_k} \right]$; in particular we have $m \in T_{\overset{\Gamma}{p}}^{X_n}$.

Suppose it is player X_{n+1} 's turn to move. In this case, we describe player X_{n+1} 's strategy. Recall $\overset{\Gamma}{p} \in T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{T^{X_{n-1}}}$, so by the usual sup-argument

$\exists m \in T_{\overset{\Gamma}{p}}^{X_{n-1}} \left[\overset{\Gamma}{p}(m) \notin \text{ORD}_{X_n}^{T^{X_{n-1}}} \right]$. The strategy for player X_{n+1} is to play such an m (i.e., to play m such that $\overset{\Gamma}{p}(m) \in T^{X_n} = T^{X_{n-1}} \setminus \text{ORD}_{X_n}^{T^{X_{n-1}}}$).

We now explain why player X_{n+1} 's strategy is a winning strategy. We have shown that any legal move by any player X_i , where $1 \leq i \leq n$, takes a position in T^{X_n} to a position in T^{X_n} and any move according to player X_{n+1} 's strategy takes a position in T^{X_n} to a position in T^{X_n} . Thus for any play $\overset{1}{y}$ according to player X_{n+1} 's strategy,

$\bar{y}(n) \in T^{X_n}$ for every n . Therefore, since $T^{X_n} \subseteq T^{X_{n-1}} \subseteq T^{X_1} \subseteq T^{X_0}$ we have

$\bar{y}(n) \notin \text{ORD}_{X_i}^{T^{X_{i-1}}}$ for $1 \leq i \leq n$. In particular, for all n , $\bar{y}(n)$ does not have $\text{ORD}_{X_i}^{T^{X_{i-1}}}$ -value

of zero for $1 \leq i \leq n$. Hence $\overset{1}{y}$ is a loss for all players $X_1, X_2, X_3, \dots, X_n$ and is therefore a win for player X_{n+1} . , (Claim (n+1))

Thus Claim 0 follows from Claims 1 through $(n + 1)$. Therefore, the game $G_{\bar{X}}^I(\bar{D})$, this canonical $(n + 1)$ -player biased open game is determined. (Theorem 6.5)

A nice result from the proof of Theorem 6.5 yields the following corollary, which uses none of the non-helping conditions of an infinite game of perfect information.

Corollary 6.6. Let $\bar{D} = (D_1, D_2, K, D_{n-1}, D_n)$ be a sequence of pairwise perpendicular sets of positions, and let $\bar{X} = (X_1, X_2, K, X_n, X_{n+1})$ be the players one through $(n + 1)$ in any order. Determined is any infinite $(n + 1)$ -player game of perfect information in which:

- (i) at most one player has a payoff set that is not open,
- (ii) at every position, there is a move m such that at the resulting position, no player other than possibly the player making the move m has a winning strategy, and
- (iii) each player is required to make such a move m .